

SYMMETRIC REPRESENTATIONS OF KNOT GROUPS

Allan L. EDMONDS* and Charles LIVINGSTON*

Department of Mathematics, Indiana University, Bloomington, IN 47401, USA

Received 15 February 1984

In this paper we give an explicit and constructive description of the pairs (μ, λ) of elements in the symmetric group S_n which can be realized as the image of the meridian-longitude pair of some knot K in the 3-sphere S^3 under a representation $\pi_1(S^3 - K) \rightarrow S_n$. The result is applied to give an otherwise nonobvious restriction on the numbers of branch curves of a branched covering of S^3 , answering a question of R. Fox and K. Perko.

AMS Subj. Class. (1980): 57M25, 57M12, 20B30, 20J06.

knot symmetric group
meridian branched covering
longitude

1. Introduction

A central technique in classical knot theory has been the study of representations of knot groups into symmetric groups. We describe here what types of such representations can occur in terms of the restriction of the representation to the 'peripheral' subgroup generated by a meridian and a longitude. In particular we solve the following problem: Given permutations μ and λ in the symmetric group S_n , when is there a knot K in the 3-sphere S^3 and a representation $\rho: \pi_1(S^3 - K) \rightarrow S_n$ such that $\rho(m) = \mu$ and $\rho(\ell) = \lambda$, where m and ℓ denote the meridian and longitude of the knot K ? The answer consists of presenting necessary and sufficient conditions that μ and λ must satisfy.

In order that a pair (μ, λ) of permutations in S_n be realizable in this way, three basic conditions must be satisfied:

- 1) *Commutativity Condition.* The commutator $[\mu, \lambda]$ is 1.
- 2) *Parity Condition.* $\lambda \in \langle \mu \rangle''$, the second commutator subgroup of the normal subgroup of S_n generated by μ . (If $\mu \neq 1$ and $n \geq 5$, $\langle \mu \rangle'' = A_n$.)
- 3) *Product Condition.* The Pontryagin product $\langle \mu, \lambda \rangle$ in $H_2(\langle \mu \rangle)$ must vanish. (This is the crucial new observation.)

* Supported in part by grants from the National Science Foundation.

The Pontryagin product is explained and computed in Section 2. The necessity of these conditions is easily proved—see Section 3. The bulk of the remainder of the paper is devoted to a proof of sufficiency.

A direct interpretation of the easy necessity result in terms of branched coverings provides an answer to a question posed by R.H. Fox and K. Perko, as stated by R. Kirby [4, Problem 1.24]: Does every simple 4-fold branched cover of a knot K in S^3 have precisely three distinct branch curves? (A *simple* 4-fold branched cover is one which corresponds to a representation $\rho: \pi_1(S^3 - K) \rightarrow S_4$ with $\rho(m)$ a transposition.) The answer is “yes” and depends upon the computation of Pontryagin products to rule out the case of two branch curves.

An outline of the remainder of this paper is as follows. In the preliminary Section 2 we give some basic definitions and notation. Some essential results about the symmetric group are described and the Pontryagin product is explained.

Section 3 contains the statement of the main theorem, the proof of necessity, and an outline of a proof of sufficiency.

Section 4 describes some particular geometric constructions and results which will be used to construct knots with prescribed types of representations to the symmetric group. In Section 5 we solve several basic special cases of the main problem. In Section 6 we present the main inductive proof.

The last Section 7 includes a discussion of branched coverings and the solution of the problem of Fox and Perko. The solution of the Fox–Perko problem in Section 7 is independent of the material in Sections 4–6.

In the appendix we prove Proposition 2.4, an algebraic result concerning the Pontryagin product in S_n .

To conclude this introduction we present two examples of knots, K_1 and K_2 in S^3 with representations $\rho: \pi_1(S^3 - K_i) \rightarrow S_5$ which illustrate the complexity of such possible representations. For readers unfamiliar with the notation of Figs. 1 and 2, a description is given in Section 2.

In Fig. 1 the trefoil knot, K_1 , is illustrated, with representation $\rho: \pi_1(S^3 - K_1) \rightarrow S_5$. In this case $\rho(m) = (1\ 2\ 3\ 4\ 5)$ and $\rho(\ell) = (1\ 2\ 3\ 4\ 5)^{-1}$. In Fig. 2 the knot K_2 is illustrated, with representation ρ satisfying $\rho(m) = (1\ 2)$, $\rho(\ell) = (3\ 4\ 5)$. By way of comparison, we note that we will be proving that it is impossible to find a knot K in S^3 with representation $\rho: \pi_1(S^3 - K) \rightarrow S_n$ with $\rho(m) = (1\ 2)(3\ 4)$ and $\rho(\ell) = (1\ 3)(2\ 4)$.

Addendum. Since completing the present manuscript we have been informed by Dennis Johnson that he found an alternative proof of the main theorem, prior to our own work. In his work he studies arbitrary groups of weight one instead of the symmetric group. The conditions of the main theorem are still necessary; he addresses the problem of finding conditions on the group G for them to be sufficient. This problem is reduced to a more algebraic one, which he then verifies for S_n and A_n , among other groups. His work does not include the specific calculations of the

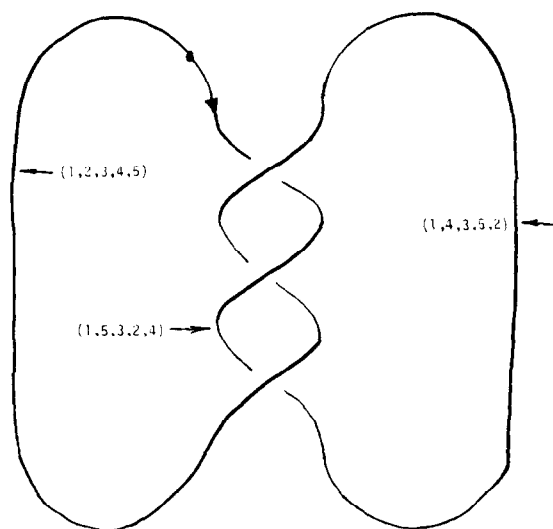


Fig. 1.

Pontryagin product for the symmetric group or the applications to branched covering spaces described here. These results are to appear in Johnson's paper, *Peripherally specified homomorphisms of knot groups*, in *Transactions of the American Mathematical Society*.

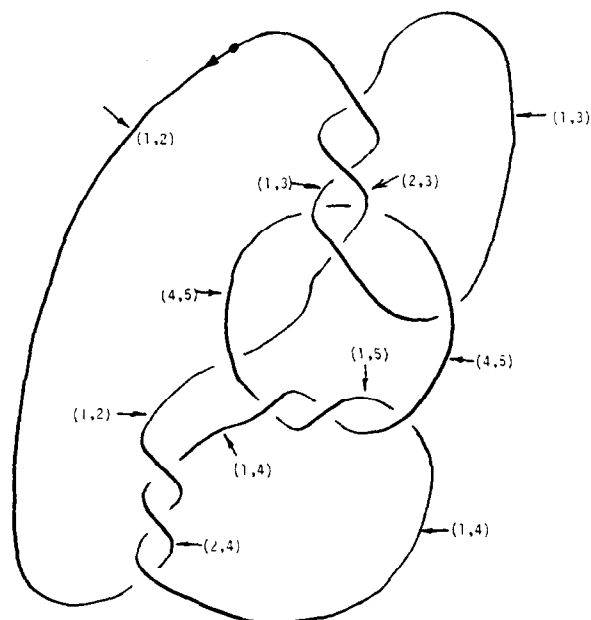


Fig. 2.

In addition, our Proposition 4.9 is a consequence of a result of González-Acuna [9]. The proof we give is essentially the same as one given by Johnson in [10].

2. Preliminaries

Knot theory

We shall work in the smooth category. A *knot* in S^3 is a connected, *oriented*, one-dimensional submanifold of S^3 . Two knots are said to be *equivalent* if they are isotopic.

Basic references for knot theory and in particular symmetric representations of knot groups are [3, 7].

Because the calculations here involve nonabelian groups, a little care must be shown in defining the meridian and longitude of a knot K . Fix a base point $x_0 \in S^3 - K$; in a knot diagram x_0 will always correspond to the nose of the observer. The boundary of a small tubular neighborhood N of K , missing x_0 is a torus T . Choose embedded curves a and b on T such that a is nullhomologous in N , b is nullhomologous in $S^3 - K$, and a and b intersect in exactly one point x_1 . Choose a path p from x_0 to x_1 . Then the *meridian* of K is the class $m = [p * a * p^{-1}]$ in $\pi_1(S^3 - K, x_0)$ and the *longitude* of K is the class $\ell = [p * b * p^{-1}]$.

The choices involved in the preceding definition affect m and ℓ by simultaneous conjugation by some element of $\pi_1(S^3 - K, x_0)$. For any pair μ, λ in S_n , however, the existence of a representation $\rho: \pi_1(S^3 - K, x_0) \rightarrow S_n$ with $\rho(m) = \mu$ and $\rho(\ell) = \lambda$ implies the existence of ρ' with $\rho'(m) = \alpha\mu\alpha^{-1}$ and $\rho'(\ell) = \alpha\lambda\alpha^{-1}$, for any $\alpha \in S_n$, simply by composing ρ with the corresponding inner automorphism of S_n . Therefore the ambiguity in the definition of m and ℓ does not affect the solution of the main problem of this paper. In addition, in most instances, we shall ignore questions concerning choice of base point.

Various representations of knot (and link) groups will be described pictorially in what follows. This is done by labeling each arc of a regular projection of the knot with a permutation. This amounts to defining a representation by prescribing its values on the set of generators of $\pi_1(S^3 - K)$ given by the set of loops which consist of a path starting at the base point above the plane of projection, running to the plane, linking a single arc of the projection in the positive direction, and then returning to the base point. Such a labeling determines a well-defined homomorphism provided the appropriate Wirtinger relation $z = yx^{-1}$ is satisfied at each crossing.

Symmetric group

We view the permutation group S_n as acting on the set $[1, n]$ of positive integers less than or equal to n . Motivated by path multiplication in the fundamental group, multiplication in S_n is done from left to right.

If $\{\gamma_i\}$ is a collection of permutations in S_n we define its *support* to be

$$\text{supp}\{\gamma_i\} = \{k \in [1, n]: \gamma_i(k) \neq k, \text{ for some } \gamma_i\}.$$

Note that $\text{supp}\{\gamma_i\} = \text{supp } H$ where H is the subgroup (not necessarily normal) generated by $\{\gamma_i\}$.

A subgroup $H \subset S_n$ acts *transitively* (on $[1, n]$) if for each $i, j \in [1, n]$ there is some $\gamma \in H$ such that $\gamma(i) = j$. A representation $\rho: G \rightarrow S_n$ is transitive if $\rho(G)$ is. A set of permutations is transitive if it generates a transitive subgroup.

We will use the following result extensively.

Proposition 2.1. *Let $\alpha, \beta \in S_n$ be commuting permutations. Then one can decompose α and β as products $\alpha = \alpha_0 \prod_{i=1}^m \alpha_i$ and $\beta = \beta_0 \prod_{i=1}^m \beta_i$ such that*

- (1) $\text{supp}(\alpha_i) \cap \text{supp}(\alpha_j) = \emptyset$ ($i \neq j$)
 $\text{supp}(\beta_i) \cap \text{supp}(\beta_j) = \emptyset$
- (2) $\text{supp}(\alpha_i) = \text{supp}(\beta_i)$ ($i \geq 1$)
- (3) $\text{supp}(\alpha_0) \cap \text{supp}(\beta_0) = \emptyset$
- (4) $\{\alpha_i, \beta_i\}$ is transitive on $\text{supp}\{\alpha_i\}$ ($i \geq 1$)
- (5) For each $i \geq 1$ there is an m_i -cycle γ_i such that $\alpha_i = \gamma_i^{a_i}$ for some a_i dividing m_i , and $\beta_i = \gamma_i \sigma_i^{b_i}$ where σ_i is some (any) cycle in the cycle decomposition of α_i .

Proof. One forms α_0 by grouping together the cycles of α which move elements of $\text{supp}(\alpha) - \text{supp}(\beta)$. Similarly one forms β_0 by grouping together the cycles of β which move elements of $\text{supp}(\beta) - \text{supp}(\alpha)$. The remainder of the decomposition is achieved by grouping the remaining cycles of α , and of β , according to the partition of $\text{supp}(\alpha) \cap \text{supp}(\beta)$ into domains of transitivity under $\{\alpha, \beta\}$. Properties (1)–(4) are immediate.

To prove (5) we may assume $\{\alpha, \beta\}$ is transitive and show there is an n -cycle γ such that $\alpha = \gamma^a(a|n)$ and $\beta = \gamma \sigma^b$ for σ a cycle of α and b some integer.

First note that α consists of n/q q -cycles and β consists of n/p p -cycles for some p, q dividing n . Otherwise, $\{\alpha, \beta\}$ could not be transitive.

Let $\alpha = (a_{11} \cdots a_{1q})(a_{21} \cdots a_{2q}) \cdots (a_{p1} \cdots a_{pq})$ and $\sigma = (a_{11} \cdots a_{1q})$.

Now β is completely determined by one of its constituent cycles. For α must act transitively (by conjugation) on the cycles of β . Moreover transitivity also implies that the labeling of the cycles of α after σ can be chosen so that one of the cycles of β is $\beta_1 = (a_{11} \ a_{21} \cdots a_{p1} \ a_{1s+1} \cdots)$ for some $s \geq 0$.

Examine $\beta' = \beta \sigma^{-s}$. Clearly $[\alpha, \beta'] = 1$. Moreover the cycle β'_1 of β' involving a_{11} is now $\beta'_1 = (a_{11} \ a_{21} \cdots a_{p1})$, since $\alpha^2(a_{11}) = a_{1s+1}$. Therefore

$$\beta' = (a_{11} \ a_{21} \cdots a_{p1})(a_{12} \cdots a_{p2}) \cdots (a_{1q} \cdots a_{pq}).$$

Set $\gamma = \beta' \sigma = \beta \sigma^{1-s}$. Then

$$\gamma = (a_{11} \ a_{21} \cdots a_{p1} \ a_{12} \cdots a_{p2} \cdots a_{1q} \cdots a_{pq})$$

and $\alpha = \gamma^p$, as required. $\square$

We shall also need a result about factoring a permutation as a product of cycles. If $\gamma \in S_n$ define $v(\gamma) = n - r$ where r is the total number of cycles (counting trivial ones) in the cycle decomposition of γ .

Proposition 2.2. *Let $\gamma \in S_n$ and $p, q \in [1, n]$. Then γ can be expressed as $\alpha\beta$ where α is a p -cycle and β is a q -cycle in S_n if and only if $v(\gamma) \geq 2n - p - q$ and $v(\gamma) \equiv 2n - p - q \pmod{2}$.*

This result is proved in [1]. We will use it only in the case when γ is a product of two disjoint cycles. The reader is invited to construct an independent proof in this case.

Pontryagin products

If g_1 and g_2 are commuting elements of a group G , then their Pontryagin product is an element $\langle g_1, g_2 \rangle \in H_2(G)$. We describe one way of defining it in a way suitable for our present purposes.

Because $[g_1, g_2] = 1$ there is a homomorphism $\phi: \mathbb{Z} \times \mathbb{Z} \rightarrow G$ given by $\phi(1, 0) = g_1$ and $\phi(0, 1) = g_2$. Now $H_2(\mathbb{Z} \times \mathbb{Z}) \approx \mathbb{Z}$ with generator z described as image of $1 \otimes 1$ under the cross product isomorphism $H_1(\mathbb{Z}) \otimes H_1(\mathbb{Z}) \rightarrow H_2(\mathbb{Z} \times \mathbb{Z})$ arising in the Künneth formula. Set $\langle g_1, g_2 \rangle = \phi_*(z)$. If g_1 and g_2 generate cyclic groups G_1 and G_2 , respectively, and $G_1 \cap G_2 = \{1\}$, then another way to view this product is via the composition $\mathcal{P}$

$$H_1(G_1) \otimes H_1(G_2) \rightarrow H_2(G_1 \times G_2) \rightarrow H_2(G),$$

where the first map is the cross-product and the second is induced by the group operation in G . Then $\langle g_1, g_2 \rangle = \mathcal{P}(\bar{g}_1 \otimes \bar{g}_2)$ where $\bar{g}_i$ is the element of $H_1(G_i) = G_i/[G_i, G_i]$ determined by g_i .

The following proposition describes two general elementary facts about the Pontryagin product. Their proofs will be left to the reader.

Proposition 2.3. *Suppose g, h, h_1, h_2 are elements of a group G with $[g, h] = [g, h_1] = [g, h_2] = 1$. Then*

- (i) $\langle g, g \rangle = 0$
- (ii) $\langle g, h_1 h_2 \rangle = \langle g, h_1 \rangle + \langle g, h_2 \rangle$
- (iii) $\langle g, h \rangle = -\langle h, g \rangle$
- (iv) $\langle kgk^{-1}, khk^{-1} \rangle = \langle g, h \rangle$ for $k \in G$.

The particular calculations we shall need are given in the following result. The proof is given in the appendix.

Proposition 2.4. (i) *For $n \geq 4$, $\langle (12), (34) \rangle \neq 0$ in $H_2(S_n)$. (ii) For $n = 6$ or 7 , $\langle (123), (456) \rangle \neq 0$ in $H_2(A_n)$.*

These two results enable one to compute the nontriviality or triviality of any such Pontryagin product for the symmetric group of alternating group. For example, in $H_2(S_n)$

$$\begin{aligned}
 \langle (12)(34), (13)(24) \rangle &= \langle (12)(34), (13)(24) \rangle + \langle (13)^2, (13)(24) \rangle \\
 &= \langle (12)(34), (13)(24) \rangle + \langle (13), (13)(24) \rangle \\
 &\quad + \langle (13), (13)(24) \rangle \\
 &= \langle (1234), (13)(24) \rangle + \langle (13), (13) \rangle + \langle (13), (24) \rangle \\
 &= \langle (1234), (1234)^2 \rangle + \langle (13), (13) \rangle + \langle (13), (24) \rangle \\
 &= \langle (13), (24) \rangle \neq 0.
 \end{aligned}$$

And in $H_2(A_n)$, $n \geq 6$,

$$\begin{aligned}
 \langle (123)(456), (123)(465) \rangle &= \langle (123)(456), (123) \rangle + \langle (123)(456), (465) \rangle \\
 &= \langle (123), (123) \rangle + \langle (456), (123) \rangle + \langle (123), (465) \rangle \\
 &\quad + \langle (456), (465) \rangle \\
 &= 0 - \langle (123), (456) \rangle + 2\langle (123), (456) \rangle + 0 \\
 &= \langle (123), (456) \rangle = 0, \quad \text{unless } n = 6 \text{ or } 7.
 \end{aligned}$$

3. Statement of main theorem

The following is the precise statement of the main result of this paper.

Theorem 3.1. *If μ and λ are elements of S_n , then there is a knot $K \subset S^3$ which has a representation $\rho: \pi_1(S^3 - K) \rightarrow S_n$ such that $\rho(m) = \mu$ and $\rho(\ell) = \lambda$ if and only if*

- (i) $[\mu, \lambda] = 1$
- (ii) $\lambda \in \langle \mu \rangle''$, and
- (iii) $\langle \mu, \lambda \rangle = 0$ in $H_2(\langle \mu \rangle)$.

Proof of necessity. Assertion (i) follows because m and ℓ commute in $\pi_1(S^3 - K)$, since they have representatives lying on a torus.

Assertion (ii) follows from the standard facts that $\pi_1(S^3 - K)$ is normally generated by m and that ℓ lies in the second commutator subgroup of $\pi_1(S^3 - K)$. This second fact follows from the observations that for any Seifert surface F for K the longitude ℓ is a product of a commutators of standard generators of $\pi_1(F)$ and that the inclusion-induced homomorphism $H_1(F) \rightarrow H_1(S^3 - K)$ is the zero map, as one verifies by considering linking numbers.

To check assertion (iii), observe first that $\rho(\pi_1(S^3 - K)) \subset \langle \mu \rangle \subset S_n$. The map $f: \mathbb{Z} \times \mathbb{Z} \rightarrow \langle \mu \rangle$ induced by restricting ρ to $\pi_1(T)$ factors as $\mathbb{Z} \times \mathbb{Z} \rightarrow \pi_1(S^3 - K) \rightarrow \langle \mu \rangle$.

Since $S^3 - K$ is an Eilenberg–MacLane space it suffices to show that $\langle m, \ell \rangle = 0$ in $H_2(S^3 - K) = 0$. More simply, $\langle m, \ell \rangle$ is represented by the peripheral torus in the knot complement which clearly bounds in $S^3 - K$. $\square$

The proof of sufficiency of conditions (i)–(iii) will be completed in Section 6, after suitable techniques are developed in Sections 4–5. In Section 4 we will discuss seven techniques for constructing representations. In Section 5 these techniques will be applied to construct four different kinds of basic representations. The sufficiency proof will be an inductive one based on a notion of complexity. It will be shown how to reduce a given problem to one of lower complexity or to one of the basic cases.

4. General constructions of representations

We will use the following terminology. An ordered pair of permutations (μ, λ) in S_n is said to be *realizable in S_n* if there is a knot K with a representation $\rho: \pi_1(S^3 - K) \rightarrow S_n$ where $\rho(m) = \mu$ and $\rho(\ell) = \lambda$. Therefore the main theorem is a description of all realizable pairs. This section describes seven techniques for constructing such realizations.

Inserting a free meridian

If μ factors as $\mu_1\mu_2$ with $\text{supp}(\mu_1) \cap \text{supp}(\mu_2) = \emptyset$ and if (μ_1, λ) is realizable where $\text{supp}(\lambda) \cap \text{supp}(\mu_2) = \emptyset$, one can often realize (μ, λ) in a trivial way.

Proposition 4.1. *If (μ_1, λ) is realizable in $S_{n_1} \subset S_n$ and $\mu_2 \in S_n$ with $\text{supp}(\mu_2) \cap \text{supp}(S_{n_1}) = \emptyset$, then $(\mu_1\mu_2, \lambda)$ is realizable in S_n .*

Proof. Let $\rho_1: \pi_1(S^3 - K) \rightarrow S_{n_1}$ realize (μ_1, λ) . Choose a regular projection of K and a corresponding Wirtinger presentation [3] of $\pi_1(S^3 - K)$, say $\langle x_1, \dots, x_s; r_1, \dots, r_s \rangle$. Here each x_i is conjugate to the meridian m and each relator r_i is of the form $x_j x_i x_j^{-1} = x_k$. Define

$$\rho: \pi_1(S^3 - K) \rightarrow S_n \text{ by } \rho(x_i) = \rho_1(x_i)\mu_2.$$

The hypotheses about μ_2 and the form of the relators show that this is a well defined representation with $\rho(m) = \mu_1\mu_2$. That $\rho(\ell) = \lambda$ follows from the additional fact that the exponent sum of a word in $\{x_i\}$ representing ℓ is zero. $\square$

Connected sums

If K_1 and K_2 are knots in S^3 , denote by $K_1 \# K_2$ their *oriented* connected sum. This operation allows one to combine factors in the longitudes.

Proposition 4.2. *If (μ, λ_1) and (μ, λ_2) are realizable in S_m , then so is $(\mu, \lambda_1\lambda_2)$.*

Proof. For $i = 1, 2$, let $\rho_i: \pi_1(S^3 - K_i) \rightarrow S_n$ realize (μ, λ_i) . It follows from Van Kampen's theorem that $\pi_1(S^3 - K_1 \# K_2) \approx \pi_1(S^3 - K_1) *_Z \pi_1(S^3 - K_2)$. We may assume base points and paths have been chosen so that each meridian m_i of K_i is the image of $1 \in \mathbb{Z}$ under the amalgamation. Since $\rho_1(m_1) = \rho_2(m_2)$, ρ_1 and ρ_2 define a representation $\rho: \pi_1(S^3 - K_1 \# K_2) \rightarrow S_n$ with $\rho(m) = \mu$. The longitude ℓ for $K_1 \# K_2$ can be described as $\ell_1 \ell_2$, the product of (the images of) the longitudes ℓ_1 and ℓ_2 of K_1 and K_2 . Therefore $\rho(\ell) = \lambda_1 \lambda_2$, as required. $\square$

Torus knots

The fundamental group of the complement of a torus knot has a simple, well known presentation which can be used to realize certain pairs of permutations.

Lemma 4.3. *The fundamental group of the complement of the (p, q) -torus knot has presentation $\langle x, y: x^p = y^q \rangle$. If r and s are integers such that $ps - qr = 1$, then*

$$m = y^s x^{-r} \quad \text{and} \quad \ell = x^p m^{-pq}.$$

Proof. The derivation of the presentation is standard. Let T be a standard torus in S^3 given as the boundary of a regular neighborhood of an unknotted oriented curve. Let K be the (p, q) curve on T . Let D be the dual (r, s) curve on T , intersecting K in one point. Let P be a curve on T , parallel to K , also meeting D in one point which we denote by x_0 and use as base point. Let M denote a meridian curve to K based at x_0 , intersecting T in only two points x_0 and x_1 , both on D .

Now $T - K$ is an open annulus separating $S^3 - K$ into two components. One is the interior open solid torus with fundamental group $\mathbb{Z}$, generated by x , and the other is the exterior open solid torus with fundamental group $\mathbb{Z}$, generated by y . The generator of the fundamental group of the annulus, represented by P , gives x^p on the inside and y^q on the outside. This describes the presentation $\langle x, y: x^p = y^q \rangle$.

The meridian M can be represented as a product of two paths, each easily computed in terms of x and y . Let a_1 be the path which follows M from x_0 to x_1 in the exterior solid torus and then follows D from x_1 to x_0 . Let a_2 be the path which follows D from x_0 back to x_1 missing K and then follows M from x_1 to x_0 in the interior solid torus. Clearly $M \approx a_1 * a_2$, rel x_0 . Now a_1 is homotopic rel end points in $S^3 - K$ to a path in the exterior component representing y^s , while a_2 is homotopic rel end points in $S^3 - K$ to a path in the interior component representing x^{-r} . Therefore M is represented by $y^s x^{-r}$.

Finally the curve P represents the longitude of K times the meridian to the pq -power. Therefore the longitude is given by $x^p m^{-pq}$. $\square$

This result can be applied to prove the following realization result.

Proposition 4.4. *Suppose $\mu \in S_n$ can be represented as $\beta\alpha$ where α is a product of disjoint p -cycles and β is a product of disjoint q -cycles, with $(p, q) = 1$. Then (μ, μ^{-pq}) is realizable.*

Proof. Let K be the (p, q) -torus knot with $\pi_1(S^3 - K) = \langle x, y: x^p = y^q \rangle$ as above. Define $\rho: \pi_1(S^3 - K) \rightarrow S_n$ by $\rho(x) = \alpha^q$ and $\rho(y) = \beta^p$. Choose integers r and s such that $ps - qr = 1$. Then by Lemma 4.3 $\rho(m) = \beta^{sp} \alpha^{-rq} = \beta \alpha$ and $\rho(\ell) = \alpha^{pq} (\beta \alpha)^{-pq} = (\beta \alpha)^{-pq}$. $\square$

It should be noted at this point that the techniques introduced so far cannot suffice to realize all allowable (μ, λ) . One can easily argue that the pair $((1234), (567))$ cannot be realized in S_7 by a torus knot or a connected sum of torus knots.

Companions

Let J be a knot in a solid torus $S^1 \times B^2$. If $f: S^1 \times B^2 \rightarrow S^3$ is an embedding with $f(S^1 \times 0)$ a nontrivial knot K and with $f(S^1 \times (1, 0))$ the longitude of K , then we say that $f(J)$ is the *satellite of K formed from J* .

In $S^1 \times B^2$ the knot J has a meridian m and longitude ℓ , in $\pi_1(S^1 \times B^2 - J)$ obtained by viewing $S^1 \times B^2$ as lying in S^3 in a standard way. In addition, in $S^1 \times S^1 = \partial(S^1 \times B^2)$ there are a meridian m' and a longitude ℓ' given by $1 \times \partial B^2$ and $S^1 \times (1, 0)$, respectively. (Choose the base point $x_0 = (1, (1, 0))$.)

Proposition 4.5. *If there is a representation $\rho: \pi_1(S^1 \times B^2 - J) \rightarrow S_n$ with $\rho(m) = \mu$, $\rho(\ell) = \lambda$, $\rho(m') = \mu'$, and $\rho(\ell') = \lambda'$, then (μ, λ) is realizable provided (μ', λ') is realizable.*

Proof. Let $K \subset S^3$ with representation $\rho': \pi_1(S^3 - K) \rightarrow S_n$ realize (μ', λ') . Let J^* be the satellite of K formed using $J \subset S^1 \times B^2$. Now $S^3 - J^*$ contains a separating torus coming from $S^1 \times S^1 \subset S^1 \times B^2$. The representations ρ and ρ' are defined on the fundamental groups of the closures of the two complementary domains. The hypotheses imply that the two representations agree on $\pi_1(S^1 \times S^1)$ and hence define the required representation $\pi_1(S^3 - J^*) \rightarrow S_n$ realizing (μ, λ) . $\square$

We now describe a procedure for constructing knots in $S^1 \times B^2$ to use in applying the previous proposition.

Proposition 4.6. *Let L be a link of two components J and J' in S^3 with J' unknotted. Let J and J' have meridians and longitudes m, ℓ and m', ℓ' , respectively, with respect to a common base point. Assume there is a representation $\rho: \pi_1(S^3 - L) \rightarrow S_n$ with $\rho(m) = \mu$, $\rho(\ell) = \lambda$, $\rho(m') = \mu'$, and $\rho(\ell') = \lambda'$. If (λ', μ') is realizable, then so is (μ, λ) .*

Proof. Because J' is unknotted, $S^3 - J'$ is an open solid torus with meridian ℓ' and longitude m' . The result now follows from Proposition 4.5. $\square$

For a sample application of this result, see Proposition 5.5.

Note that in the situation of Proposition 4.6 the Pontryagin products $\langle \mu, \lambda \rangle$ and $\langle \mu', \lambda' \rangle$ are necessarily equal, since the two peripheral tori are cobordant.

Band sums

A represented link with two components having conjugate meridian classes in S_n can often be altered via band sum to produce a new represented link. We have to keep track of the newly created longitudes as well.

Proposition 4.7. *Let L be a link in S^3 containing components K_1 and K_2 (and perhaps others). Suppose there is a representation $\rho: \pi_1(S^3 - L) \rightarrow S_n$ with $\rho(m_1) = \mu_1$ and $\rho(m_2) = \mu_2$, where $m_i \in \pi_1(S^3 - L)$ is the meridian to K_i . Assume that $\mu_1 = \sigma \mu_2 \sigma^{-1}$ with $\sigma \in \rho \pi_1(S^3 - L) \subset S_n$. It is possible to form a new link L' by connecting K_1 and K_2 by a band and to define a new representation $\rho': \pi_1(S^3 - L') \rightarrow S_n$ with $\text{Image } \rho' = \text{Image } \rho$ and $\rho'(m'_1) = \mu_1$, where m'_1 is the meridian to the band sum of K_1 and K_2 .*

Proof. Let $x_0 \in S^3 - L$ denote the base point and x_1 and x_2 denote the chosen points near K_1 and K_2 used to define m_1 and m_2 . Let p_1 and p_2 denote the chosen paths from x_0 to x_1 and x_2 , respectively. The hypotheses imply that it is possible to choose an embedded path g from x_1 to x_2 in $S^3 - L$ such that $\rho[p_1 * g * p_2] = \sigma$. To form the band sum, extend g to a path connecting K_1 to K_2 ; then extend g to an embedding $G: I \times I \rightarrow S^3$ such that $G(I \times I) \cap L = G(I \times 0 \cup I \times 1)$ and $G(I \times I) \cap K_1 = G(I \times 0)$ and $G(I \times I) \cap K_2 = G(I \times 1)$. Choose G so that orientations are consistent. Let $B = G(I \times I)$. See Fig. 3. Set $K'_1 = (K_1 \cup K_2 - B \cap (K_1 \cup K_2)) \cup G(\partial I \times I)$, and $L' = (L - K_1 \cup K_2) \cup K'_1$, a new oriented link.

To define $\rho': \pi_1(S^3 - L') \rightarrow S_n$ first define $\rho'': \pi_1((S^3 - (L \cup B)) \rightarrow S_n$ to be the composition of the inclusion-induced surjection $\pi_1(S^3 - (L \cup B)) \rightarrow \pi_1(S^3 - L)$ with ρ . Now L' is obtained from $L \cup B$ by removing $G((0, 1) \times I)$ from $L \cup B$ and adding it to $S^3 - (L \cup B)$. In particular $S^3 - L'$ is obtained from $S^3 - L \cup B$ by adding a 2-handle along a curve which in $S^3 - L$ is homotopic to $a_1 * g * a_2^{-1} * g^{-1}$ where $m_i = [p_i * a_i * p_i^{-1}]$. Under ρ , this class, after being suitably connected to x_0 , maps

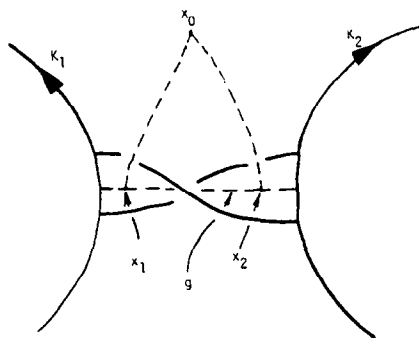


Fig. 3.

to 1 in S_n :

$$\begin{aligned} p_1 * a_1 * g * a_2^{-1} * g^{-1} * p_1^{-1} \\ \simeq p_1 * a_1 * p_1^{-1} * p_1 * g * p_2^{-1} * p_2 * a_2^{-1} * p_2^{-1} * p_2 * g^{-1} * p_1^{-1}, \end{aligned}$$

which maps to $\mu_1 \sigma \mu_2^{-1} \sigma^{-1} = 1$. Therefore ρ'' factors through $\pi_1(S^3 - L')$ to yield the required representation ρ' . $\square$

Addendum 4.8. The image $\rho'(\ell')$ where ℓ' is the longitude of K'_1 (using the path p_i from x_0 to x_1) is $\mu_1^{-2k} \lambda_1 \sigma \lambda_2 \sigma^{-1}$ where $k = \text{link}(K_1, K_2)$.

Proof. One choice of longitude ℓ' (not the preferred one, in general) is describable in $S^3 - L$ as

$$q = p_1 * b_1 * g * b_2 * g^{-1} * p_1^{-1},$$

where $\ell_1 = p_i * b_i * p_i^{-1}$. Now $\rho[q] = \lambda_1 \sigma \lambda_2 \sigma^{-1}$. But this curve q has linking number $2k$ with K'_1 . Therefore $\rho'(\ell') = \mu_1^{-2k} \lambda_1 \sigma \lambda_2 \sigma^{-1}$, as required. $\square$

Making a representation surjective

The band sum operation can be used to made the image of a representation as large as possible.

Proposition 4.9. Given $\mu \in S_n$, there is a knot $K \subset S^3$ with meridian m and a representation $\rho: \pi_1(S^3 - K) \rightarrow S_n$ such that $\rho(m) = \mu$ and $\rho(\pi_1(S^3 - K)) = \langle \mu \rangle$, the normal subgroup generated by μ , unless $n = 4$ and μ is conjugate to $(12)(34)$.

Proof. First consider the case when μ is an odd permutation, so $\langle \mu \rangle = S_n$. Form a link in S^3 consisting of unknotted and unlinked components in one-to-one correspondence with the conjugates of μ in S_n . Then $\pi_1(S^3 - L)$ is free with basis consisting of the meridians of the link. Define $\rho': \pi_1(S^3 - L) \rightarrow S_n$ by mapping each meridian to the corresponding conjugate of μ . By Proposition 4.7 L can be transformed by band sums into a knot K with representation $\rho: \pi_1(S^3 - K) \rightarrow S_n$ with $\rho(m) = \mu$ and $\text{Image } \rho = \text{Image } \rho' = \langle \mu \rangle = S_n$.

When $\mu \in A_n$ perform the same construction, starting with a link with components in one-to-one correspondence with the conjugates of μ in A_n . If $n \neq 4$, $\langle \mu \rangle = A_n$ and the construction produces the desired representation. If $n = 4$ and μ is a 3-cycle, again $\langle \mu \rangle = A_4$, and the construction works.

Finally if $n = 4$ and $\mu = (12)(34)$, $\langle \mu \rangle \approx \mathbb{Z}_2 \times \mathbb{Z}_2$. Any $\rho: \pi_1(S^3 - K) \rightarrow S_4$ with $\rho(m) = \mu$ has image in $\mathbb{Z}_2 \times \mathbb{Z}_2$ and hence factors through $\mathbb{Z}$, the abelianization of $\pi_1(S^3 - K)$. Thus $\text{Image } \rho$ is cyclic of order 2 in this case. $\square$

Corollary 4.10. Give $\mu \in S_n$, $\mu \neq 1$, there is a knot K with meridian m and representation

$\rho: \pi_1(S^3 - K) \rightarrow S_n$ with $\rho(\mu) = \mu$ and Image ρ equal to S_n or A_n according to whether μ is odd or even, except when $n = 4$ and μ is conjugate to (12)(34).

Corollary 4.11. *If (μ, λ) is realizable, in S_n , then it is realizable using a knot and representation mapping onto S_n or A_n (depending on the parity of μ), except when $n = 4$ and μ is conjugate to (12)(34).*

Proof. By Corollary 4.10 some (μ, λ') is realizable in the required way. Connected sum of sufficiently many of these realizes $(\mu, 1)$ in this way. Then connected sum with an arbitrary realization of (μ, λ) completes the construction. $\square$

Reversers

We describe one last technique for constructing new representations out of old ones.

Proposition 4.12. *Suppose (μ, λ) is realizable in S_n and that there is a $\tau \in \langle \mu \rangle \subset S_n$ such that $\tau\mu\tau^{-1} = \mu^{-1}$. Then the pair $(\mu, \mu^{2r}\tau^2\lambda)$ is realizable for some integer r .*

Proof. If $\mu = (12)(34)$ in S_4 the only candidates for τ have $\tau^2 = 1$, for which the result is trivial. Avoiding this case we know by the preceding result that we may assume that (μ, λ) is realized by a representation $\rho: \pi_1(S^3 - K) \rightarrow S_n$ with Image $\rho = \langle \mu \rangle$.

In $S^3 - K$ choose a knot K_1 such that $\rho[K_1] = \tau$ (K_1 suitably connected to the base point x_0). Let N_1 be a tubular neighborhood of K_1 in $S^3 - K$ and $N_2 \subset \text{int } N_1$ be a slightly smaller tubular neighborhood. Let $K_2 \subset \partial N_2$ be a (1, 2) curve.

Now there is a representation $\rho_2: \pi_1(N_1 - K_2) \rightarrow S_n$ taking the longitude of N_1 to τ and the meridian of N_1 to 1 (as required by ρ), and the meridian of K_2 to μ . Note that the natural longitude of K_2 in N_1 , homologous in $N - K_2$ to two longitudes of N_1 , maps to $\mu^{-2}\tau^2$ under ρ . See Fig. 4.

These data define a representation $\rho_3: \pi_1(S^3 - (K \cup K_2)) \rightarrow S_n$ with the meridians of K and K_2 mapping to μ . Now perform the band sum as in Proposition 4.7 with $\sigma = 1$. By Addendum 4.8 this realizes $(\mu, \mu^{-2k}\tau^2\lambda)$ where $k = \text{link}(K_2, K)$ as required. $\square$

We shall show early in the next section that any (μ, μ^{2k}) is realizable, thereby increasing the usefulness of this result.

5. Basic representations

In this section we will begin to apply the techniques of Section 4 to realize certain basic pairs of permutations by representations of knots.

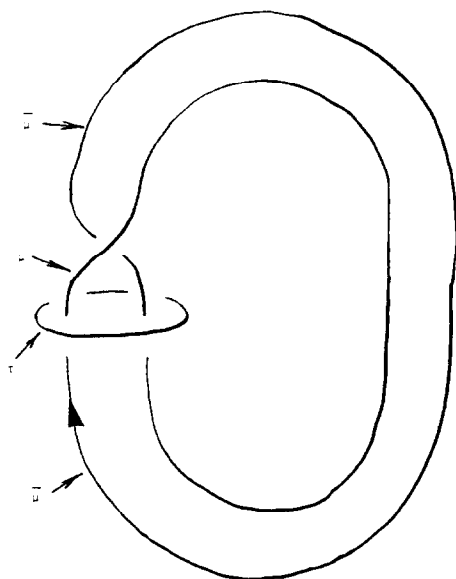


Fig. 4.

Proposition 5.1. Let $\alpha, \beta \in S_n$, where α is a k -cycle and $\text{supp}(\alpha) \cap \text{supp}(\beta) = \emptyset$. Suppose the pair $(\alpha\beta, \alpha^2)$ satisfies the parity, commutativity, and product conditions. Then $(\alpha\beta, \alpha^2)$ is realizable in S_n .

Comment. The three conditions only place restrictions in this case when $n \leq 7$: parity when $n = 3$ or 4 and the $\mathbb{Z}_3$ Pontryagin product when $n = 6$ or 7 and α and β are 3-cycles.

Proof. The analysis will involve a case by case discussion.

Case 1. $k \geq 4$. Then (α, α^2) satisfies all required conditions in S_k . We shall show (α, α^2) is realizable in S_k . Then $(\alpha\beta, \alpha^2)$ is realizable in S_n by Proposition 4.1 (inserting a free meridian).

Subcase 1.a. k odd. One can write $\alpha = \gamma\delta$ where γ is a $(k-2)$ -cycle and δ is a product of two disjoint 2-cycles. Since $(2, k-2) = 1$, the $(2, k-2)$ torus knot can be used to realize $(\alpha, \alpha^{-2(k-2)})$, by Proposition 4.4. Since $-(k-2)$ is prime to k , a suitable connected sum of copies of this knot provides the desired realization, by Proposition 4.2. Note that in this case, additional connected sums realize $(\alpha\beta, \alpha)$ as well.

Subcase 1.b. k even. Then $\alpha = \gamma\delta$ where γ is a $(k-1)$ -cycle and δ is a 2-cycle. Now $(2, k-1) = 1$ so the $(2, k-1)$ torus knot realizes $(\alpha, \alpha^{-2(k-1)})$ and since $-(k-1)$ is prime to k , connected sums yield (α, α^2) .

Case 2. $k = 2$. This case is trivial since $\alpha^2 = 1$.

Case 3. $k = 3$. This is the most complicated case since parity and product conditions can fail.

Subcase 3.a. $n = 3$ or 4 . Up to conjugacy this is $\alpha = (123)$, $\beta = 1$. But $((123), (132))$ fails to satisfy the parity condition.

Subcase 3.b. $n = 5$. Up to conjugacy there are two pairs to consider: $((123), (132))$ and $((123)(45), (132))$.

For the first write $(123) = (12534)(14)(35)$, a 5-cycle times a product of disjoint 2-cycles. Therefore the $(2, 5)$ torus knot realizes $((123), (123)^{-10}) = ((123), (132))$.

To realize $((123)(45), (132))$ one would like just to insert (45) , but it is *not* 'free' in S_5 . Therefore we use another torus knot. Write $(123)(45) = (12345)(14)$. Then the $(2, 5)$ torus knot realizes $((123)(45), ((123)(45))^{-10}) = ((123)(45), (132))$.

Subcase 3.c. $n = 6$. Realize $((123), (132))$ and $((123)(45), (132))$ as in S_5 . The only remaining case up to conjugacy is $((123)(456), (132))$ which violates the product condition.

Subcase 3.d. $n = 7$. Up to conjugacy there are five conceivable pairs. Realize $((123), (132))$ and $((123)(45), (132))$ in S_5 . The second one can be used to realize $((123)(45)(67), (132))$ by inserting the free meridian (67) .

The case $((123)(456), (132))$ is ruled out by the product condition.

Finally realize $((123)(4567), (132))$ in S_7 using a $(2, 7)$ torus knot: $(123)(4567) = (1234567)(14)$. Proposition 4.4 shows $((123)(4567), ((123)(4567))^{-14}) = ((123)(4567), (123)(4567))$ is realizable. Connected sum of eight of these yields the desired pair.

Subcase 3.e. $n \geq 8$.

Subcase 3.e.i. $\# \text{supp}(\alpha\beta) \leq n - 2$. Realize (α, α^2) in S_5 , as in Subcase 3.b., and then insert the free meridian β using Proposition 4.1.

Subcase 3.e.ii. β factors as $\beta_1\beta_2$ with $\text{supp}(\beta_1) \cap \text{supp}(\beta_2) = \emptyset$ and β_1 a 2-cycle. Realize $(\alpha\beta_1, \alpha^2)$ in S_5 by Subcase 3.b., and insert the free meridian β_2 .

Subcase 3.e.iii. β factors as $\beta_1\beta_2$ with $\text{supp}(\beta_1) \cap \text{supp}(\beta_2) = \emptyset$ and β_1 a 4-cycle. Realize $(\alpha\beta_1, \alpha^2)$ in S_7 by Subcase 3.d., and insert the free meridian β_2 .

Subcase 3.e.iv. β factors as $\beta_1\beta_2$ with $\text{supp}(\beta_1) \cap \text{supp}(\beta_2) = \emptyset$ and β_1 an m -cycle, $m > 4$. Then $\alpha\beta_1$ factors as a p -cycle times a q -cycle with $(p, q) = 1$ and $(3, p) = (3, q) = 1$. To do this choose $n/2 + 1 \leq p, q \leq n$ satisfying these three requirements—say $q = p + 1$ where $p \equiv 1 \pmod{3}$ (if n is odd) or $q = p + 2$ where p is odd and $p \equiv 2 \pmod{3}$ (if n is even)—and apply Proposition 2.2. Then using a torus knot one realizes $(\alpha\beta_1, (\alpha\beta_1)^{pq})$. Using a connected sum of this with itself, if necessary, yields $(\alpha\beta_1, \alpha\beta_1^r)$ for some r . Since $\alpha \in A_n$, either $\beta_1 \in A_n$ or r is even. By Subcases 1.a and 1.b one can therefore realize (β_1, β_1^{-r}) in S_m , and hence $(\alpha\beta_1, \beta_1^{-r})$ in $S_{m+3} \subset S_n$. Then a connected sum realizes $(\alpha\beta_1, \alpha)$.

Subcase 3.e.v. The cycle decomposition of β is nontrivial and consists only of 3-cycles. The process of inserting free meridian factors reduces us to two final cases, up to conjugacy: $((123)(456), (132))$ in S_8 and $((123)(456)(789), (132))$ in S_9 .

Since $((123), (132))$ can be realized in S_5 by Subcase 3.b., $((123)(456), (132))$ can be realized in S_8 by inserting the free meridian (456) .

In S_9 , $((123)(456)(789), (132))$ can be realized if $((123)(456), (123)(456))$ can be realized in S_6 : For then by inserting free meridians and conjugating one can realize

(μ, β_i) where $\mu = (123)(456)(789)$ and $\beta_1 = (123)(456)$, $\beta_2 = (456)(789)$, and $\beta_3 = (123)(789)$. Then $(\mu, \beta_1\beta_2^2\beta_3) = ((123)(456)(789), (132))$ can be realized by connected sums.

It remains to realize $((123)(456), (123)(456))$ in S_6 . Write $(123)(456) = (12346)(14)(56)$. Then the $(2, 5)$ torus knot realizes $((123)(456), (123)^{10}(456)^{10}) = ((123)(456), (123)(456))$. $\square$

Corollary 5.2. *If $\mu \in S_n$, then (μ, μ^2) is realizable in S_n , unless $n = 3$ or 4 and μ is a 3-cycle.*

Proof. Express μ as a product $\mu_1 \cdots \mu_r$ of disjoint cycles. Then $\mu^2 = \mu_1^2 \cdots \mu_r^2$. By Proposition 5.1 one can realize (μ, μ^2) with the following exceptions: (i) $n = 3$ or 4 and μ is a 3-cycle (this we have ruled out by hypothesis) and (ii) $n = 6$ or 7 and μ is a product of two disjoint 3-cycles.

Therefore connected sums imply that (μ, μ^2) is realizable except possibly for when $\mu = (123)(456)$ in S_6 or S_7 . But in this case (μ, μ) was realized at the very end of the last subsubcase 3.e.v in the proof of Proposition 5.1. Therefore by connected sum (μ, μ^2) is realizable in this case as well. $\square$

Corollary 5.3. (Squares of reversers). *Suppose (μ, λ) is realizable in S_n and that there is a $\tau \in \langle \mu \rangle \subset S_n$ such that $\tau\mu\tau^{-1} = \mu^{-1}$. Then $(\mu, \tau^2\lambda)$ is also realizable in S_n .*

Proof. By Proposition 4.12 $(\mu, \mu^{2r}\tau^2\lambda)$ is realizable for some integer r . By Corollary 5.2 (μ, μ^2) , and hence by connected sums (μ, μ^{-2r}) , is realizable. Then connected sums show that $(\mu, \tau^2\lambda)$ is realizable, as required. $\square$

One needs to be able to realize pairs of the form (μ, μ) (assuming the parity condition is satisfied). The simplest case which cannot be derived from Proposition 5.1 is that where μ is a product of an even number of even length cycles. The following result addresses this situation.

Proposition 5.4. *Let $\mu = \alpha_1\alpha_2\beta$ and $\lambda = \alpha_1\alpha_2$ where α_1 and α_2 are disjoint, nontrivial, even length cycles in S_n , $n \geq 5$, and β is a permutation in S_n disjoint from $\alpha_1\alpha_2$. Then $(\alpha_1\alpha_2\beta, \alpha_1\alpha_2)$ is realizable in S_n .*

Proof. Let α_1 and α_2 have even lengths m_1 and m_2 , respectively.

If $m_1 + m_2 \geq 6$, then there are always two distinct odd primes p and q in the interval $[(m_1 + m_2)/2, m_1 + m_2]$, by Bertrand's hypothesis [6]. In this case one can write $\alpha_1\alpha_2$ as the product of a p -cycle and a q -cycle in $S_{m_1+m_2}$: by Proposition 2.2 $v(\alpha_1\alpha_2) = m_1 + m_2 - 2$ is even, and $m_1 + m_2 - 2 \geq 2(m_1 + m_2) - p - q$, since p and q are distinct primes greater than or equal to $(m_1 + m_2)/2$. Then by Proposition 4.4 the (p, q) torus knot realizes $(\alpha_1\alpha_2, (\alpha_1\alpha_2)^{-pq})$ in $S_{m_1+m_2}$. Since pq is relatively prime to m_1m_2 (any odd divisor of m_i is at most $m_i/2$ which is less than $(m_1 + m_2)/2$), $\alpha_1\alpha_2$

can be expressed as a power of $(\alpha_1\alpha_2)^{-pq}$. Therefore a connected sum of these knots realizes $(\alpha_1\alpha_2, \alpha_1\alpha_2)$ in $S_{m_1+m_2}$. Finally the free meridian β can be inserted by Proposition 4.1 to realize $(\alpha_1\alpha_2\beta, \alpha_1\beta_2)$.

It remains to handle the more complicated case $m_1 + m_2 = 4$, that is $m_1 = m_2 = 2$. It suffices to realize pairs of the form $((12)(34)\beta, (12)(34))$.

In S_4 $((12)(34), (12)(34))$ is not realizable since it fails to satisfy the parity condition.

In S_5 $((12)(34), (12)(34))$ can be realized. Write $(12)(34) = (12345)(153)$. Then by Proposition 4.4. The $(3, 5)$ torus knot realizes the pair $((12)(34), (12)(34))$.

For $n \geq 6$ the process of inserting free meridians reduces one to the case of realizing $((12)(34)\beta, (12)(34))$ where β is a k -cycle ($k \geq 2$), in S_{k+4} .

If k is odd, write $\mu = (12)(34)\beta$ as a product of a p -cycle and a q -cycle where p and q are odd and relatively prime, using Proposition 2.2: Now $v(\mu) = k + 4 - 3 = k + 1$ and is even; therefore we require p, q both odd, $p + q \geq k + 7$; by Bertrand's hypothesis we may choose two primes in the interval $((k + 5)/2, k + 5)$, provided $(k + 5)/2 \geq 4$ (except for $k = 5$; but $(5, 10)$ contains relatively prime integers $p = 7, q = 9$).

Now, using the (p, q) torus knot one realizes $((12)(34)\beta, (12)(34)\beta^{-pq})$. For $k \geq 5$ we realize (β, β^{pq}) in S_k by Corollary 5.2 and connected sums, since k is odd. Inserting $(12)(34)$ realizes $((12)(34)\beta, \beta^{pq})$ in S_{k+4} , and connected sum realizes $((12)(34)\beta, (12)(34))$. If $k = 3$ we can realize $((34)\beta, \beta)$ in S_{k+2} , by Proposition 5.1, which yields $((34)\beta, \beta^2)$, and connected sum. Now insert a free meridian to achieve $((12)(34)\beta, \beta)$. Connected sum of $((12)(34)\beta, (12)(34)\beta^{-pq})$ with pq of these yields the required pair.

If k is even, the square of reverser technique works in all cases. Let σ have order 2 such that $\sigma\beta\sigma^{-1} = \beta^{-1}$. Then $\tau = (1423)\sigma$ is a reverser for $\mu = (12)(34)\beta$ and $\tau \in \langle \mu \rangle = S_n$. By Corollary 5.3 $((12)(34)\beta, \tau^2) = ((12)(34)\beta, (12)(34))$ is realizable. $\square$

Recall from Section 2 that commuting permutations μ, λ decompose as $\mu = \mu_0 \prod \mu_i$ and $\lambda = \lambda_0 \prod \lambda_i$ with $\text{supp}(\mu_0) \cap \text{supp}(\lambda_0) = \emptyset$. We now explain how to realize basic pairs with $\lambda_0 \neq 1$. We shall use the technique of companions from Section 4.

Proposition 5.5. *If μ is a k -cycle and λ is a 3-cycle in S_n with $\text{supp}(\mu) \cap \text{supp}(\lambda) = \emptyset$, then (μ, λ) is realizable in S_n unless $n = 6$ or 7 and μ is a 3-cycle.*

Proof. If k is even or if $k \equiv 1 \pmod{4}$ or $k + 3 \leq n - 2$, μ has a reverser σ of order 2 disjoint from λ , $\sigma \in \langle \mu \rangle$. Set $\tau = \sigma\lambda^2$. Then τ is a reverser for μ with $\tau^2 = \lambda$. Therefore (μ, λ) is realizable by Corollary 5.3.

It remains to handle the case $k \equiv 3 \pmod{4}$ with $k + 3 \geq n - 1$. (The first such case is $k = 7, n = 11$.) We use companions to reduce such a case to one already considered.

Let K be a knot with representation ρ to S_n realizing $(\mu, 1)$. By Proposition 4.9 we may assume $\text{Image } \rho = A_n$.

Consider the three-component link illustrated in Fig. 5. The representation is given by that for K , extended as indicated.

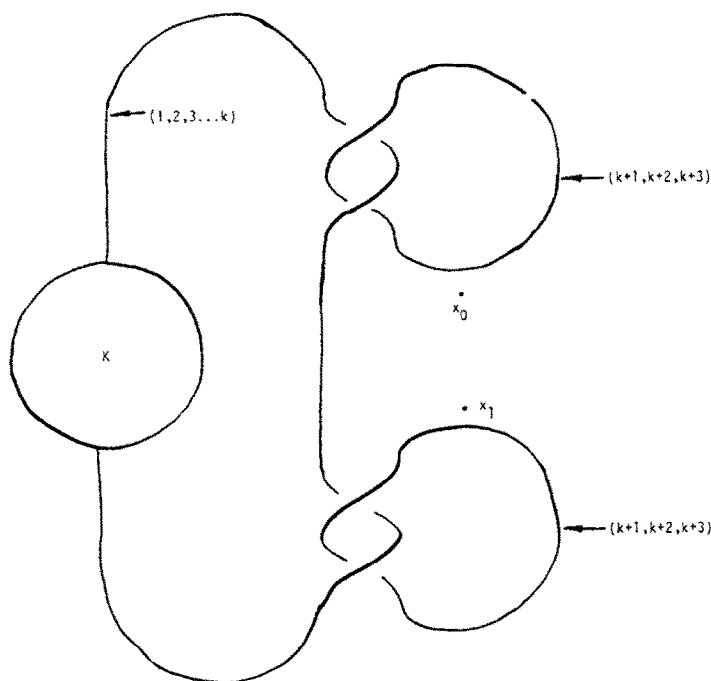


Fig. 5.

Since $\text{Image } \rho = A_n$, there is a path c from x_0 to x_1 such that c followed by the straight line path from x_1 to x_0 is represented to $(12)(34)$. Form the band sum of the two right hand components along c . The diagram in Fig. 6 is a schematic description of the resulting link. In particular note that the right hand component is unknotted. The meridian-longitude pair for the left hand component is $((12 \cdots k), (k+1, k+2, k+3)^2)$; that for the right hand component is $((k+1, k+2, k+3), (12 \cdots k)(12)(34)) = ((k+1, k+2, k+3), (245 \cdots (k-1)k))$ with longitude class a $(k-2)$ -cycle. By Proposition 4.6 the desired pair is realizable if the pair $(\mu_1, \lambda_1) = ((245 \cdots (k-1)k), (k+1, k+2, k+3))$ is realizable in S_n . Since μ_1 is a $(k-2)$ -cycle this pair is realizable by square of reverser as indicated above. $\square$

Corollary 5.6. *If $\mu, \lambda \in S_n$ with $\mu \neq 1$, $\text{supp}(\mu) \cap \text{supp}(\lambda) = \emptyset$, and $\lambda \in A_n$, then (μ, λ) is realizable except if $n = 6$ or 7 , μ is a 3-cycle, and λ is a 3-cycle.*

Proof. The excluded cases fail to satisfy the product condition. Recall that A_n is generated by 3-cycles. Assume first either that $n \geq 8$ or that μ is not a 3-cycle. Write λ as a product $\lambda_1 \cdots \lambda_r$ of 3-cycles. By Proposition 5.5 realize each (μ, λ_i) . By connected sums realize (μ, λ) .

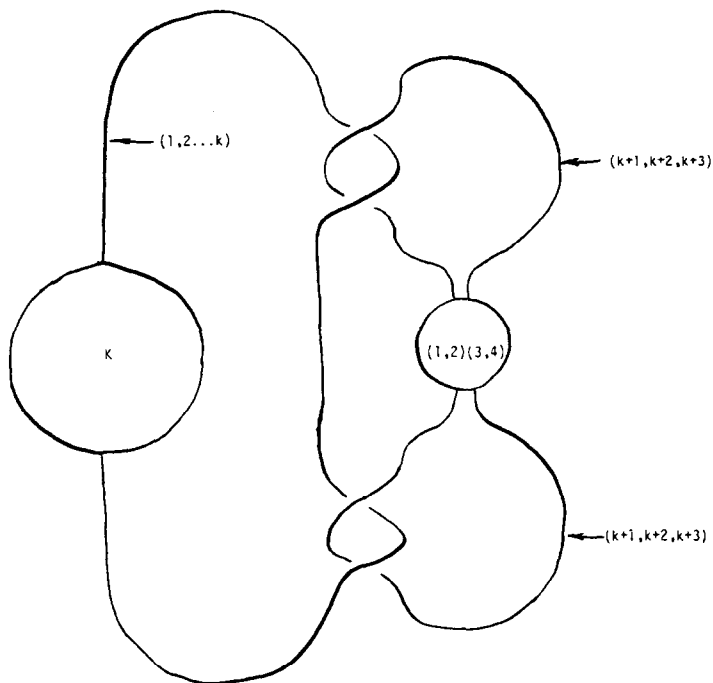


Fig. 6.

It remains to realize $((123), (45)(67))$. The permutation $\tau = (12)(4657)$ is a reverser for $\mu = (123)$ with $\tau \in \langle \mu \rangle$ and $\tau^2 = (45)(67)$. Therefore $((123), (45)(67))$ is realizable by Corollary 5.3. $\square$

We next address the more delicate problem of realizing an *odd* permutation as a longitude class disjoint from a meridian class.

Lemma 5.7. *Let (μ, λ) be a pair of permutations in S_n satisfying the commutativity, parity, and product conditions. Let $\mu = \mu_0 \mu'$ and $\lambda = \lambda_0 \lambda'$ where $\text{supp}(\mu') = \text{supp}(\lambda')$, $\text{supp}(\lambda_0) \cap \text{supp}(\mu) = \text{supp}(\mu_0) \cap \text{supp}(\lambda) = \emptyset$ (compare Proposition 2.1). If λ_0 is an odd permutation, then the cycle decomposition of μ contains two nontrivial cycles of the same length.*

Proof. Let $\mu = \mu_1 \cdots \mu_k$ be the cycle decomposition of μ and suppose the μ_i have distinct lengths. Then since $[\mu, \lambda] = 1$, λ has the form $\lambda = \lambda_0 \mu_1^{p_1} \cdots \mu_k^{p_k}$. We may suppose there is a number e such that μ_i has even length for $i \leq e$ and odd length for $e < i \leq k$. Then the parity condition implies $\lambda \in A_n$ and since λ_0 is odd, we have $\sum_{i=1}^e p_i$ odd. The Pontryagin product $\langle \mu, \lambda \rangle$ is easily computed to be $\sum_{i=1}^e p_i(e-1) + e \pmod{2}$. Since $\sum_{i=1}^e p_i \equiv 1 \pmod{2}$, the product condition becomes $e-1+e \equiv 0 \pmod{2}$. This contradiction means that the hypothesis that the cycles of μ have distinct lengths is wrong. $\square$

Proposition 5.8. *Let μ_1 and μ_2 be disjoint k -cycles and λ_0 be a transposition disjoint from μ_1 and μ_2 in S_n , $n = 2k + 2$. Then some pair of the form $(\mu_1\mu_2, \lambda_0\lambda_1)$ is realizable in S_n where λ_1 is a $2k$ -cycle with $\lambda_1^2 = \mu_1\mu_2$.*

Proof. For convenience let $\mu_1 = (12 \cdots k)$, $\mu_2 = (k+1, k+2, \dots, 2k)$, and $\lambda_0 = (2k+1, 2k+2)$. We use a companionship argument similar in spirit to that used before for Proposition 5.5. Start with a knot K and representation $\pi_1(S^3 - K) \rightarrow S_n$ realizing $(\mu_1\mu_2, 1)$ and onto A_n , which we can do by Proposition 4.9.

Consider the link illustrated in Fig. 7. The rectangles indicate numbers of full twists to insert in the indicated bands. The oval indicates a band that winds through K along a path which when closed to a loop through the base point above the plane of the diagram yields the class $(2, k+1)\lambda_0 \in A_n$. A direct calculation shows that the meridian-longitude pair for the left hand component is $(\mu_1\mu_2, \lambda_0\lambda_1)$, which we wish to realize. A calculation shows that the meridian-longitude pair for the right hand component is $((12)\lambda_0, \lambda')$ where λ' is disjoint from $(12)\lambda_0$ and is nontrivial provided $k > 2$. By Corollary 5.6 the pair $(\lambda', (12)\lambda_0)$ can be realized. Therefore by Proposition 4.5, $(\mu_1\mu_2, \lambda_0\lambda_1)$ can be realized, if $k > 2$.

If $k = 2$, consider the companion diagram in Fig. 8. It shows that $((12)(34), (1324)(56))$ is realizable, provided $((1625)(34), (12)(56))$ is. But the latter pair can be realized by Proposition 5.1. $\square$

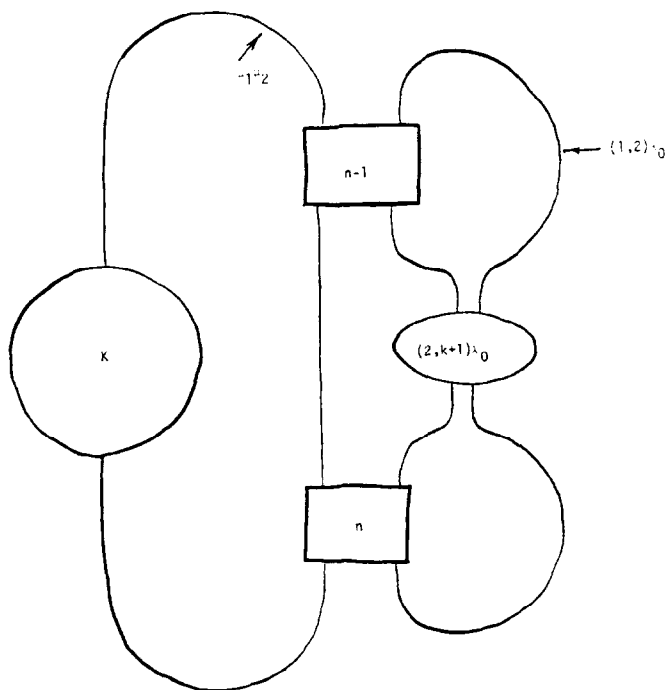


Fig. 7

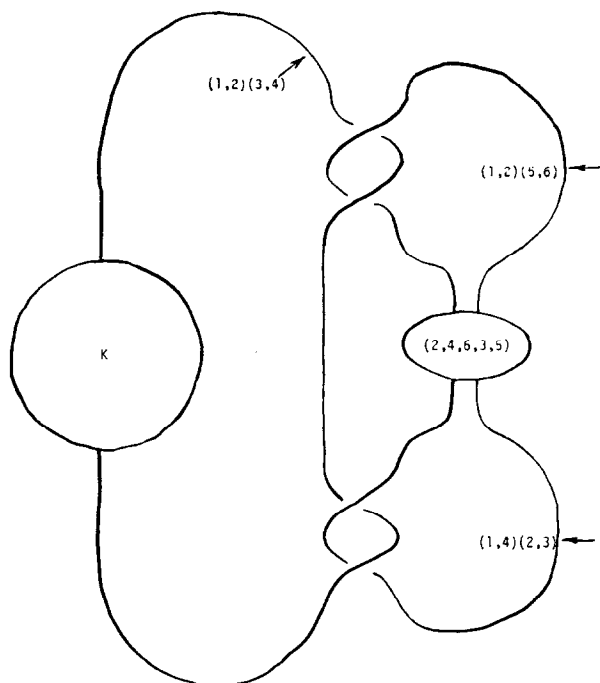


Fig. 8.

6. Proof of the main theorem

Let (μ, λ) be an ordered pair of permutations in S_n satisfying the commutativity, parity, and product conditions. This section is devoted to the proof that (μ, λ) can be realized in S_n . The proof has a number of cases and subcases, and proceeds by induction on the complexity of (μ, λ) , which we now define.

Since $[\mu, \lambda] = 1$ we can write $\mu = \mu_0 \mu'$ and $\lambda = \lambda_0 \lambda'$ where $\text{supp}(\mu') = \text{supp}(\lambda')$ and $\text{supp}(\mu_0) \cap \text{supp}(\mu') = \text{supp}(\lambda_0) \cap \text{supp}(\lambda') = \text{supp}(\mu_0) \cap \text{supp}(\lambda_0) = \emptyset$. We define the *complexity* of (μ, λ) in S_n to be

$$c(\mu, \lambda) = (n, \# \text{supp}(\lambda_0), \# \lambda').$$

These triples lie in $(\mathbb{Z}_+)^3$ which we order lexicographically for purposes of mathematical induction.

To begin the induction we observe that the results of Section 5 show immediately that (μ, λ) is realizable if $n \leq 5$. We now let (μ, λ) be given and assume that any (μ^*, λ^*) in S_m satisfying the commutativity, parity, and product conditions and $c(\mu^*, \lambda^*) < c(\mu, \lambda)$ is realizable in S_m .

Case 1. $\# \text{supp}(\lambda_0) > 0$.

Subcase 1.A. $\text{sign}(\lambda_0) = 0$. By Corollary 5.6 (μ, λ_0) is realizable unless $n = 6$ or 7 and μ is a 3-cycle. But such pairs (μ, λ) do not satisfy the working hypotheses.

Now it follows from the technique of connected sums that $(\mu, \lambda_0 \lambda')$ is realizable if and only if (μ, λ') is realizable. But $c(\mu, \lambda') < c(\mu, \lambda)$, so this subcase follows by induction.

Subcase 1.B. $\text{sign}(\lambda_0) = 1$. First note that μ is not a 3-cycle, since commutativity and parity then imply $\text{sign}(\lambda_0) = 0$. Choose λ'_0 with $\text{supp}(\lambda'_0) \subset \text{supp}(\lambda_0)$ such that $\text{sign}(\lambda'_0) = 0$ and $\lambda'_0 \lambda_0$ is a transposition. By Corollary 5.6 (μ, λ'_0) is realizable. Therefore connected sums imply $(\mu, \lambda_0 \lambda_1)$ is realizable if and only if $(\mu, \tau \lambda_1)$ is, where $\tau = \lambda'_0 \lambda_0$ is a transposition. (We have not necessarily reduced complexity, since λ_0 may be a transposition.)

By Lemma 5.7 the cycle decomposition of μ contains two cycles of the same length. Therefore Proposition 5.8 and the process of inserting free meridians imply that some $(\mu, \tau \lambda'_1)$ with $\text{supp}(\lambda'_1) \subset \text{supp}(\mu)$ is realizable. Then connected sums imply $(\mu, \tau \lambda_1)$ is realizable if and only if $(\mu, \lambda_1 \lambda'_1)$ is realizable. Note that $c(\mu, \lambda_1 \lambda'_1) = (n, 0, *) < c(\mu, \lambda)$. Therefore induction completes the proof in Subcase 1.B and, hence, Case 1.

Case 2. $\# \text{supp}(\lambda_0) = 0$, i.e. $\lambda_0 = 1$. This case is rather more complicated and will involve more cases and subcases. First recall that by Proposition 2.1 we can write $\mu = \mu_0 \prod_{i=1}^k \gamma_i^{a_i}$ and $\lambda = \prod_{i=1}^k \gamma_i \sigma_i^{b_i}$ where γ_i is a cycle of length ℓ_i and $a_i | \ell_i$, $1 \leq a_i \leq \ell_i$, σ_i is a component of $\gamma_i^{a_i}$ of length $m_i = \ell_i / a_i$, and $\text{supp}(\mu_0) \cap \text{supp}(\lambda) = \emptyset$.

Note that the pair $(\mu_0^{-1} \mu, \lambda)$ satisfies all the necessary conditions in S_n , $r = \# \text{supp}(\mu_0^{-1} \mu)$. The method of inserting free meridians shows that if $(\mu_0^{-1} \mu, \lambda)$ is realizable, in S_r , then (μ, λ) is realizable in S_n . Henceforth we therefore assume $\mu_0 = 1$.

Subcase 2.A. Some $a_i \geq 3$. For convenience assume $a_1 \geq 3$. Set $m = m_1 = \ell_1 / a_1$. By a conjugation we may assume that $\mu = (12 \dots m)(m+1, \dots, 2m)(2m+1, \dots, 3m)\mu^*$, where $\text{supp}(\mu^*) \cap [1, 3m] = \emptyset$ and $\lambda(1) = m+1$.

Consider the link illustrated in Fig. 9. A direct computation shows that

$$\lambda_L = (1, m+1, 2m+1)(2, m+2, 2m+2) \dots (m, 2m, 3m)$$

and

$$\lambda_R = (1, m+1, 2m+1)^{2m+1}(2, m+2, 2m+2) \dots (m, 2m, 3m)$$

where λ_L and λ_R are the longitudes of the left and right components, respectively. The method of companions combined with inserting free meridians now shows that (μ, λ_L) is realizable if $(\lambda_R, (1, m+1, 2m+1))$ is realizable.

To see that $(\lambda_R, (1, m+1, 2m+1))$ is realizable, first suppose that $m+1$ is prime to 3. Then $c(\lambda_R, (1, m+1, 2m+1)) < c(\mu, \lambda)$. Moreover the commutativity, parity, and product conditions are all satisfied. Therefore $(\lambda_R, (1, m+1, 2m+1))$ is realizable by induction, and hence (μ, λ_L) is realizable in this case.

If $3|m+1$, the complexity has actually increased (a free longitude factor has arisen). In this case Corollary 5.6 implies directly that $(\lambda_R, (1, m+1, 2m+1))$ is realizable. (The excluded cases $n = 6$ or 7 and λ_R a 3-cycle cannot occur here because

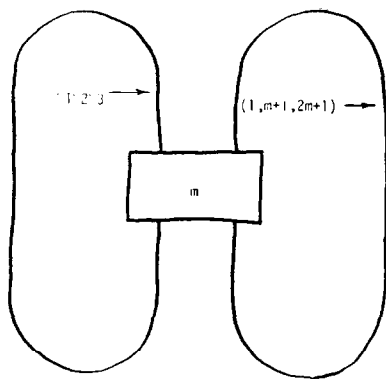


Fig. 9. $\rho_1 = (1, 2 \cdots m)$, $\rho_2(m+1 \cdots 2m)$, $\rho_3 = (2m+1 \cdots 3m)$.

then the original problem (μ, λ) would have been conjugate to $((12)(34)(56), (135)(246))$ which is ruled out by the $\mathbb{Z}_3$ product condition.)

Thus in any case (μ, λ_L) can be realized. Now the method of connected sums shows that the desired (μ, λ) is realizable if and only if $(\mu, \lambda\lambda_L^{-1})$ is realizable. Further, all necessary conditions are satisfied for the latter pair and $c(\mu, \lambda\lambda_L^{-1}) < c(\mu, \lambda)$ since $\lambda\lambda_L^{-1}$ now has a fixed point, so $\text{supp}(\lambda\lambda_L^{-1}) \subsetneq \text{supp}(\lambda)$. Induction implies $(\mu, \lambda\lambda_L^{-1})$ is realizable, completing the proof in this case.

Subcase 2.B. Two a_i 's equal 1 (and all $a_i \leq 2$). Say $a_1 = a_2 = 1$. In this case $\mu = \gamma_1\gamma_2\gamma_3^{a_3} \cdots$ and $\lambda = \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3\sigma_3^{b_3} \cdots$, since $\sigma_1 = \gamma_1$ and $\sigma_2 = \gamma_2$ here.

Subcase 2.B.i. γ_1 or γ_2 of odd length. Suppose γ_1 has odd length. By Proposition 5.1 (μ, γ_1^2) can be realized unless γ_1 is a 3-cycle and $\mu = \gamma_1\gamma_2$ is a product of two 3-cycles. In the latter case Corollary 5.2 and the process of inserting a free meridian shows $(\mu, \gamma_1^2\gamma_2^2)$ is realizable. Therefore, by connected sums either (μ, γ_1) or $(\mu, \gamma_1\gamma_2)$ can be realized. Let (μ, δ) stand for one of these.

Then (μ, λ) is realizable if and only if $(\mu, \lambda\delta^{-1})$ is realizable. The pair $(\mu, \lambda\delta^{-1})$ satisfies the necessary conditions and $c(\mu, \lambda\delta^{-1}) < c(\mu, \lambda)$ since $\text{supp}(\lambda\delta^{-1}) \subsetneq \text{supp}(\lambda)$. Therefore induction applies to show that $(\mu, \lambda\delta^{-1})$, and hence (μ, λ) , is realizable.

Subcase 2.B.ii. γ_1 and γ_2 of even length. Proposition 5.4 implies that $(\mu, \gamma_1\gamma_2)$ is realizable. Now proceed as in the preceding case.

Subcase 3. $k = 1$, i.e., $\mu = \mu_0\gamma_1^{a_1}$ (and $a_1 \leq 2$). If $a_1 = 1$, we are seeking to realize $(\mu_0\gamma_1, \gamma_1^{b_1})$. Parity implies that b_1 is even if γ_1 has even length. Then by Proposition 5.1, (γ_1, γ_1^2) is realizable and $(\gamma_1, \gamma_1^{b_1})$ is realizable from this by connected sums.

The remaining possibility in this subcase, that $a_1 = 2$, can be ruled out. We would be seeking $(\mu_0\gamma_1^2, \gamma_1\sigma_1^{b_1})$. Now γ_1 has even length, so b_1 is odd and σ_1 also has even length by the parity condition. Now compute the Pontryagin product $\langle \mu_0\gamma_1^2, \gamma_1\sigma_1^{b_1} \rangle = \langle \mu_0, \gamma_1\sigma_1^{b_1} \rangle + \langle \gamma_1^2, \gamma_1 \rangle + \langle \gamma_1^2, \sigma_1^{b_1} \rangle = 0 + 0 + 1 \not\equiv 0 \pmod{2}$.

Subcase 4. $k = 2$, $a_1 = 1$, and $a_2 = 2$ (or equivalently $a_1 = 2$ and $a_2 = 1$). In this case we wish to realize $(\gamma_1\gamma_2^2, \gamma_1^{b_1}\gamma_2\sigma_2^{b_2})$. If γ_1 has odd length, then we can realize $(\gamma_1, \gamma_1^{b_1})$

and use connected sums to reduce complexity. So assume γ_1 has even length; γ_2 also has even length, since the term γ_2^2 appears. But then the Pontryagin product

$$\begin{aligned} \langle \gamma_1 \gamma_2^2, \gamma_1^{b_1} \gamma_2 \sigma_2^{b_2} \rangle &= \text{sign}(\gamma_2 \sigma_2^{b_2}) + \text{sign}(\sigma_2) \text{sign}(\sigma_2^{b_2}) \\ &= \text{sign}(\gamma_2) + b_2 \text{sign}(\sigma_2) + b_2 \text{sign}(\sigma_2)^2 \\ &= \text{sign}(\gamma_2) + 2b_2 \text{sign}(\sigma_2) \\ &\equiv \text{sign}(\gamma_2) \\ &\equiv 1 \pmod{2}, \end{aligned}$$

contradicting the product condition.

Subcase 5. At least two a_i 's equal 2. Suppose $a_1 = a_2 = 2$. Then we wish to realize $\mu = \gamma_1^2 \gamma_2^2 \gamma_3^{a_3} \cdots$ and $\lambda = \gamma_1 \sigma_1^{b_1} \gamma_2 \sigma_2^{b_2} \gamma_3 \sigma_3^{b_3} \cdots$. We may suppose $\gamma_1^2 = (1, \dots, m_1)(m_1 + 1, \dots, 2m_1)$ and $\gamma_2^2 = (2m_1 + 1, \dots, 2m_1 + m_2)(2m_1 + m_2 + 1, \dots, 2m_1 + 2m_2)$. Further we may assume $\gamma_1 \sigma_1^{b_1}(1) = m_1 + 1$ and $\gamma_2 \sigma_2^{b_2}(2m_1 + 1) = 2m_1 + m_2 + 1$.

We shall show that one can realize $(\gamma_1^2 \gamma_2^2, \tilde{\lambda})$ in $S_{2m_1 + 2m_2}$, where $\text{supp}(\tilde{\lambda}) \subset \text{supp}(\gamma_1^2 \gamma_2^2)$ and either $\tilde{\lambda}(m_1 + 1) = 1$ or $\tilde{\lambda}(2m_1 + m_2 + 1) = 2m_1 + 1$. Given this, $(\mu, \tilde{\lambda})$ is realizable by inserting free meridians. Then by connected sums (μ, λ) is realizable if and only if $(\mu, \lambda \tilde{\lambda})$ is realizable. But $(\mu, \lambda \tilde{\lambda})$ has reduced complexity since products are preserved, $\text{supp}(\lambda \tilde{\lambda}) \subsetneq \text{supp}(\lambda)$, and no free longitude cycles are introduced. Therefore induction will complete the proof of Subcase 5 and hence the theorem. There are two subcases.

Subcase 5.A. At least one of m_1 and m_2 is odd. We assume m_1 is odd. We use the link depicted in Fig. 10. Here $\tau_1 = (1, m_1 + 1)$ is a transposition which overlaps the two m_1 -cycles of γ_1^2 . One checks directly that this assignment determines a well defined representation to S_n .

To compute the respective longitudes, let $\tau_2, \dots, \tau_{m_1}$ be the successive conjugates of τ_1 by powers of γ_1^2 . Then the left hand longitude is $\lambda_L = \tau_1 \cdots \tau_{m_1} \gamma_2^{m_1}$. Note that $\lambda_L \in A_n$ and that $\lambda_L(1) = m + 1$. To compute the required Pontryagin product we

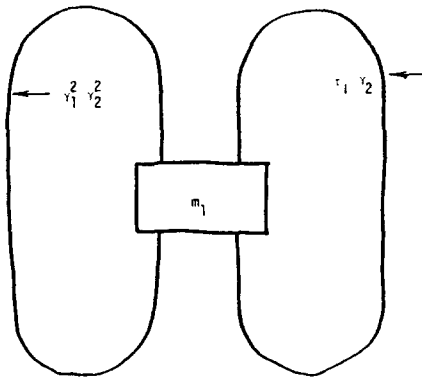


Fig. 10.

note that $\lambda_L = \gamma_1 \bar{\sigma} \gamma_2^{m_1}$ where σ is one of the two m_1 -cycles of γ_1^2 . Then $\langle \gamma_1^2 \gamma_2^2, \lambda_2 \rangle = \langle \gamma_1^2 \gamma_2^2, \gamma_1 \bar{\sigma} \gamma_2^{m_1} \rangle = 0$ since m_1 is odd and $n \geq 10$.

Similarly the right hand longitude is $\lambda_R = \tau_1^{m_1+1} \tau_2 \cdots \tau_q (\gamma_2^2)^{m_1} = \tau_2 \cdots \tau_q \gamma_2^{2m_1}$. Therefore the desired $(\gamma_1^2 \gamma_2^2, \lambda_L)$ can be realized if $(\tau_2 \cdots \tau_q \gamma_2^{2m_1}, \tau_1 \gamma_2)$ can be realized. Note that there is a disjoint transposition here, so complexity has gone up. However, by Proposition 5.8 $(\tau_2 \tau_3, \tau_1 \lambda_1)$ is realizable in S_6 where λ_1 is a 4-cycle with $\lambda_1^2 = \tau_2 \tau_3$. By inserting free meridians we realize $(\tau_2 \cdots \tau_q \gamma_2^{2m_1}, \tau_1 \lambda_1)$. The method of connected sums then says that the desired pair is realizable if and only if $(\tau_2 \cdots \tau_q \gamma_2^{2m_1}, \lambda_1 \gamma_2)$ is realizable. Now $\lambda_1 \gamma_2$ contains no cycle disjoint from $\tau_2 \cdots \tau_q \gamma_2^{2m_1}$ and $\# \text{supp}(\lambda_1 \gamma_2) < \# \text{supp}(\gamma_1 \sigma_1^b \gamma_2 \sigma_2^b) = n$. Therefore complexity has now been reduced. Induction implies that $(\tau_2 \cdots \tau_q \gamma_2^{2m_1}, \lambda_1 \gamma_2)$ is realizable, as required.

Subcase 5.B. Both m_1 and m_2 even. The preceding construction fails because a nontrivial Pontryagin product is introduced. So consider the link illustrated in Fig. 11. Here

$$\begin{aligned} \mu_1 &= \gamma_1^2 \gamma_2^2 = (1, \dots, m_1)(m_1 + 1, \dots, 2m_1) \\ &\quad \times (2m_1 + 1, \dots, 2m_1 + m_2)(2m_1 + m_2 + 1, \dots, 2m_1 + 2m_2) \end{aligned}$$

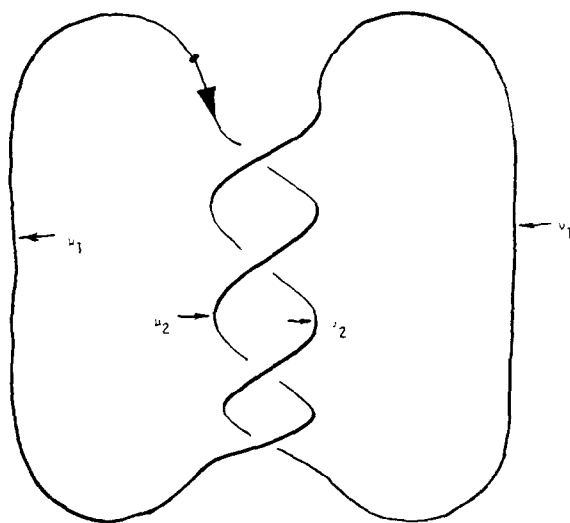


Fig. 11.

and

$$\begin{aligned} \nu_1 &= (1, m_1 + 1, 3, m_1 + 3, \dots) \\ &\quad \times (2m_1 + 1, 2m_1 + m_2 + 1, 2m_1 + 3, 2m_1 + m_2 + 3, \dots); \end{aligned}$$

so

$$\begin{aligned} \mu_2 &= \nu_1^{-1} \mu_1 \nu_1 = (m_1 + 1, 2, m_1 + 3, 4, \dots)(1, m_1 + 2, \dots) \\ &\quad \times (2m_1 + m_2 + 1, 2m_1 + 2, \dots)(2m_1 + 1, 2m_1 + m_2 + 2, \dots) \end{aligned}$$

and

$$\begin{aligned}\nu_2 &= \mu_2^{-1} \nu_1 \mu_2 = (m_1 + 2, 2, m_1 + 4, r, \dots) \\ &\quad \times (2m_1 + m_2 + 2, 2m_1 + 2, 2m_1 + m_2 + 4, \dots).\end{aligned}$$

One checks that this gives a well defined representation, i.e. $\nu_2^{-1} \mu_2 \nu_2 = \mu_1$ and $\mu_1^{-1} \nu_2 \mu_1 = \nu_1$.

The left hand longitude is

$$\begin{aligned}\lambda_L &= \nu_1 \nu_2 = (1, m_1 + 1, 3, m_1 + 3)(m_1 + 2, 2, m_1 + r, 4, \dots) \\ &\quad \times (2m_1 + 1, 2m_1 + m_2 + 1, \dots)(2m_1 + m_2 + 2, 2m_1 + 2, \dots);\end{aligned}$$

the right hand longitude is

$$\lambda_R = \mu_2 \mu_1 = (1, m_1 + 3, 5, m_1 + 7, \dots) \cdots$$

We must check that the Pontryagin products $\langle \mu_1, \lambda_L \rangle = \langle \nu_1, \lambda_R \rangle$ vanish. Only the $\mathbb{Z}_2$ product matters, since $n \geq 8$ here. Now $\lambda_L = \gamma_1 \sigma_1^{c_1} \gamma_2 \sigma_2^{c_2}$ for some c_1 and c_2 by Proposition 2.1(5), where

$$\begin{aligned}\gamma_1 &= (1, m_1 + 1, 2, m_1 + 2, \dots), \\ \gamma_2 &= (2m_1 + 1, 2m_1 + m_2 + 1, 2m_1 + 2, 2m_1 + m_2 + 2, \dots), \\ \sigma_1 &= (1, 2, \dots, m_1), \quad \text{and} \quad \sigma_2 = (2m_1 + 1, \dots, 2m_1 + m_2).\end{aligned}$$

To compute c_1 and c_2 , note that $(\gamma_1 \sigma_1^{c_1})(m_1 + 1) = \sigma_1^{c_1}(2) = 2 + c_1 \bmod m_1$, and $\lambda_L(m_1 + 1) = 3$. Therefore $c_1 = 1$, and similarly $c_2 = 1$. It follows that

$$\begin{aligned}\langle \mu_1, \lambda_L \rangle &= \langle \gamma_1^2 \gamma_2^2, \gamma_1 \sigma_1 \gamma_1 \sigma_1 \gamma_2 \sigma_2 \rangle \\ &= \langle \gamma_1^2, \sigma_1 \rangle + \langle \gamma_2^2, \sigma_2 \rangle \\ &= \text{sign}(\sigma_1) + \text{sign}(\sigma_2) \\ &= 1 + 1 \\ &= 0 \bmod 2\end{aligned}$$

since m_1 and m_2 are both even.

We wish to realize (μ_1, λ_L) . By the method of companions it suffices to realize (λ_R, ν_1) . Now the latter pair has decreased complexity since $\text{supp}(\nu_1) \subset \text{supp}(\gamma_1)$ and $\text{supp}(\gamma_1) \subsetneq \text{supp}(\lambda_L)$ and $\text{supp}(\lambda_L) \subset \text{supp}(\lambda)$, where λ is as at the beginning of Subcase 5. Therefore by induction (λ_R, ν_1) is realizable, and the proof is complete. $\square$

7. Branched coverings

Let K be a knot in S^3 with meridian m and longitude ℓ , and let $\rho: \pi_1(S^3 - K) \rightarrow S_n$ be a representation. In a standard way ρ determines an n -fold branched covering

(corresponding to the covering of $S^3 - K$ with fundamental group $\rho^{-1}(S_{n-1})$, when ρ is transitive) $p: (M, B) \rightarrow (S^3, K)$. We discuss the possibilities for $(p|B)$. In particular we determine the possibilities for the number of components of B in terms of $\mu = \rho(m)$. One consequence is the resolution of a question of Fox and Perko as described below.

The subgroup of S_n generated by $\rho(m)$ and $\rho(\ell)$ acts on $\{1, 2, \dots, n\}$. Let $\circ$ be the number of orbits of this action. Let b be the number of components of B .

Lemma 7.1. $\circ = b$.

Proof. This follows from standard covering space considerations. $\square$

Theorem 7.2. If $\mu = \rho(m)$ is a k -cycle, then $b \equiv n + 1 + k \pmod{2}$.

Proof. Let $\lambda = \rho(\ell)$. Since $[\mu, \lambda] = 1$, Proposition 2.1 implies that $\lambda = \mu^q \lambda_1 \cdots \lambda_r$, where $\lambda_1, \dots, \lambda_r$ are disjoint cycles, disjoint from μ . Here we are including 1-cycles. Also $\mu^q = \mu_1 \cdots \mu_s$, where $\mu_1, \dots, \mu_s$ are disjoint cycles and $s = (k, q)$. Each μ_i has length k/s .

Clearly $\text{length}(\mu) + \sum_{i=1}^r \text{length}(\lambda_i) = n$. Since $\text{sign}(\lambda) = 0$, it follows that $\sum_{i=1}^s (\text{length}(\mu_i) - 1) + \sum_{i=1}^r (\text{length}(\lambda_i) - 1) \equiv 0 \pmod{2}$. Therefore $\text{length}(\mu) + \sum_{i=1}^r \text{length}(\lambda_i) \equiv r + s \pmod{2}$. It follows that $n \equiv r + s \pmod{2}$, and, since $\circ = r + 1 = b$, that $b \equiv n + s + 1 \pmod{2}$.

Finally we show that $s \equiv k \pmod{2}$. If k is odd, then since s divides k , s is odd also. If k is even, then the mod 2 Pontryagin product $\langle \mu, \lambda \rangle \equiv 0 \pmod{2}$ yields

$$\langle \mu, \lambda \rangle = \langle \mu, \mu^q \lambda_1 \cdots \lambda_r \rangle \equiv 0 \pmod{2}.$$

Hence $\text{sign}(\lambda_1 \cdots \lambda_r) = 0$ and, therefore, $\text{sign}(\mu^q) = 0$. This implies $q \equiv 0 \pmod{2}$. Since $s = (q, k)$ and q and k are even, $s \equiv 0 \pmod{2}$, and $s \equiv k \pmod{2}$, as required. $\square$

Fox and Perko asked whether for simple representations (μ a transposition) to S_4 , the branch set of the corresponding branched cover contains exactly 3 curves. This follows at once from Theorem 7.2. Applying the theorem with $n=4$, $k=2$, $s=(k, q)$ yields $b \equiv 1 \pmod{2}$. Clearly $b \leq 4$. On the other hand, the group generated by μ and λ will have at least two orbits, so $b = \circ \geq 2$. Therefore $b = 3$.

(An alternate proof based directly on Theorem 3.1 runs as follows: Suppose $\mu = (12)$ in S_4 . Since $[\lambda, \mu] = 1$, $\lambda = (12)^i (34)^j$. Since parity requires $\lambda \in S_4'' \subset A_4$, either $\lambda = 1$ or $\lambda = (12)(34)$. But $\langle (12), (12)(34) \rangle \equiv 1 \pmod{2}$. Therefore $\lambda = 1$. Covering space considerations now show $b = 3$.)

There is a converse to Theorem 7.2.

Theorem 7.3. If μ is a k -cycle in S_n , $1 < k < n$, and b is an integer with $b \equiv n + 1 + k \pmod{2}$, then there is a realizable pair (μ, λ) for which any corresponding branched covering has b branch curves if and only if $2 \leq b \leq n + 1 - k$, and $b \neq 2$ if

$n = 6$ and $k = 3$. (If $k = n$ then, $b = 1$; if $k = 1$, clearly $b = n$ with no true branching; and $b \neq 2$ if $n = 6$ and $k = 3$.)

Proof. Since $k < n$, we have $b \geq 2$ and $b \leq n - k + 1$. Given such a b let λ be an r cycle disjoint from μ , where $r = n - k + 2 - b$. (Since $2 \leq b \leq n - k + 1$, $1 \leq r \leq n - k$.)

We claim that (μ, λ) is realizable unless $n = 6$, $k = 3$, $b = 2$ or $n = 7$, $k = 3$, $b = 3$.

Clearly $[\mu, \lambda] = 1$. Since $b \equiv n + 1 + k \pmod{2}$, $r \equiv 1 \pmod{2}$ and $\lambda \in A_n$. Therefore $\lambda \in \langle \mu \rangle^n$ for $n \geq 5$. One checks this for $n \leq 4$ case by case. Since r is odd the $\mathbb{Z}_2$ Pontryagin product vanishes. Similarly the $\mathbb{Z}_3$ product vanishes except for $n = 6$, $k = 3$, $b = 2$ and $n = 7$, $k = 3$, $b = 3$.

The only way $n = 6$, $k = 3$, $b = 2$ could occur would be for $(\mu, \lambda) = ((123), (456))$ up to conjugacy. This is ruled out by the $\mathbb{Z}_3$ product. For $n = 7$, $k = 3$, $b = 3$ we rechoose λ : If $\mu = (123)$, set $\lambda = (45)(67)$. This pair is then realizable by Theorem 3.1. $\square$

Appendix: Proof of Proposition 2.4

Proposition 2.4. i) For $n \geq 4$, $\langle (12), (34) \rangle \neq 0$ in $H_2(S_n)$. ii) For $n = 6$ or 7 , $\langle (123), (456) \rangle \neq 0$ in $H_2(A_n)$.

Remarks. For $n \geq 4$, $H_2(S_n) \approx \mathbb{Z}_2$ and the inclusion $S_n \subset S_{n+1}$ induces an isomorphism $H_2(S_n) \rightarrow H_2(S_{n+1})$. See for example [5]; $H_2(S_n) = 0$ for $n < 4$. Similarly $H_2(A_n) \approx \mathbb{Z}_2$ for $n \geq 4$, $n \neq 6$ or 7 ; but $H_2(A_n) \approx \mathbb{Z}_6$ for $n = 6$ or 7 . We shall not use this much information.

The result of Proposition 2.4 is in some sense well known, going back to Schur [8]. But we do not know a suitable source easily accessible to geometric topologists, including especially the result on A_n . We therefore provide a sketch of the proof in several steps.

We shall show that $H_2(S_4) \approx \mathbb{Z}_2$, generated by the product class $\langle (12), (34) \rangle$ and that stabilization $H_2(S_4) \rightarrow H_2(S_n)$ is injective. Similarly we shall show that $H_2(A_n; \mathbb{Z}_3) \approx \mathbb{Z}_3$ for $n = 6$ or 7 , generated by $\langle (123), (456) \rangle$ and that $\langle (123), (456) \rangle$ dies in $H_2(A_8)$.

For $n \geq 4$ consider the inclusion $i: S_2 \times S_2 \rightarrow S_n$ with generators (12) and (34) . The following result will suffice to prove the weak stability we require.

Lemma 2.5. The image of $i^*: H^2(S_n; \mathbb{Z}_2) \rightarrow H^2(S_2 \times S_2; \mathbb{Z}_2) = (\mathbb{Z}_2)^3$ contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. View H^2 as counting central extensions. We must produce two nonsplit, inequivalent extensions of $\mathbb{Z}_2$ by $S_2 \times S_2$ which extend to extensions of $\mathbb{Z}_2$ by S_n .

One such extension is obtained by pulling back via the surjective homomorphism $S_n \rightarrow \mathbb{Z}_2$ the extension

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 1.$$

When restricted to $S_2 \times S_2$ this extension splits over (12)(34) but *not* over (12) or over (34).

To find another such we use a geometric approach. There is a permutation representation

$$S_n \rightarrow SO(n+1)$$

given by permuting the first n basis vectors and sending the $(n+1)$ st e_{n+1} to $\pm e_{n+1}$, according to the sign of the permutation.

For $n \geq 2$, $\pi_1(SO(n+1)) \approx \mathbb{Z}_2$ and we have the universal double covering

$$\mathbb{Z}_2 \rightarrow \text{Spin}(n+1) \rightarrow SO(n+1).$$

Pulling back this extension to S_n yields an extension

$$1 \rightarrow \mathbb{Z}_2 \rightarrow E_n \rightarrow S_n \rightarrow 1.$$

To complete the proof of the lemma it suffices to show that E_n does not split over (12), (34), or (12)(34). An element of $\text{Spin}(n+1)$ can be understood as a pair $(x, [\lambda])$ where $x \in SO(n+1)$ and λ is a path from 1 to x . For each x there are two possible paths since $\pi_1(SO(n+1)) \approx \mathbb{Z}_2$. The question is for which $\mathbb{Z}_2$ in $SO(n+1)$ does the induced extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$ split?

If $x \in SO(n+1)$ has order 2 we may assume, after a conjugation, that

$$x = \begin{pmatrix} -I_r & 0 \\ 0 & I_s \end{pmatrix},$$

with $r+s = n+1$ and $r \equiv 0 \pmod{2}$. The order of $(x, [\lambda])$ is either 2 or 4, independent of the choice of λ .

Claim. $(x, [\lambda])$ has order 4 if and only if $r \equiv 2 \pmod{4}$.

Now $(x, [\lambda])^2 = (1, [\lambda^2])$. In $SO(2) \subset SO(n+1)$ consider a standard path λ from 1 to

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The path λ^2 then represents a generator of $\pi_1(SO(n+1))$.

In general x lies in $SO(2) \times \cdots \times SO(2) \subset SO(n+1)$ ($r/2$ factors). Choose the path λ from 1 to x to be given by $r/2$ coordinates of the above path. Then λ^2 in $SO(n+1)$ represents the $r/2$ power of the generator of $\pi_1(SO(n+1))$.

Then for (12) $\in S_n$ the induced sequence does not split since

$$(12) \rightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & & & & \\ 1 & 0 & & & & \\ \hline & & I_{n-2} & & 0 & \\ 0 & & & 0 & & -1 \end{array} \right) \sim \begin{pmatrix} -I_2 & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

Similarly, over (34) the induced sequence does not split.

But also over (12)(34) the induced extension does not split:

$$(12)(34) \rightarrow \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 1 & & 0 & & \\ 0 & 0 & 1 & 0 & & & & \\ \hline & & & & 1 & & & \\ & & & & & \ddots & & \\ & 0 & & & & & 1 & \end{array} \right) \begin{pmatrix} -I_2 & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

In what follows we shall need the following result. Let G be any finite group with p -Sylow subgroup G_p . Let M be any G -module (hence a G_p -module). Then

$$H^n(G; M)_{(p)} \approx H^n(G_p; M)^G.$$

Here the left hand term is the p -primary part of $H^n(G; M)$, and the right hand term denotes the G -invariant elements of $H^n(G_p; M)$. If G_p is normal in G , then of course G acts on G_p by conjugations and hence acts on cohomology. When G_p is not normal this term is to be interpreted as the set of $v \in H^n(G_p; M)$ such that for all $g \in G$ the elements $g^*(v) \in H^n(gG_pg^{-1}; M)$ and v restrict to the same element of $H^n(G_p \cap gG_pg^{-1}; M)$. The proof of (2.7) is an application of the transfer; see [2; III.10].

We first apply this when $G = S_4$. For $p \neq 2$, G_p is cyclic (or trivial), and $H^3(G_p; \mathbb{Z}) = 0$. Therefore $H^3(G)_{(p)} = 0$. Universal coefficients thus implies $H_2(G; \mathbb{Z}_p) = 0$, and $H_2(G)$ is 2-primary.

The 2-Sylow subgroup G_2 of $G = S_4$ is the dihedral group of order 8. One way of viewing G_2 is an extension of $S_2 \times S_2$ generated by (12) and (34) by $\mathbb{Z}_2$ represented by (1324).

Lemma 2.8 *The inclusion $S_2 \times S_2 \rightarrow G_2$ induces an isomorphism $H_2(S_2 \times S_2) \rightarrow H_2(G_2)$.*

Proof. Let $E_{ij}^2 = H_i(\mathbb{Z}_2; \{H_j(S_2 \times S_2)\})$ be the E^2 term of the Lyndon–Hochschild–Serre spectral sequence for the extension described above. (The braces simply indicate that $H_j(S_2 \times S_2)$ is to be viewed as a $\mathbb{Z}_2$ -module.) The only nonzero term of total degree 2 is $E_{02}^2 = H_2(S_2 \times S_2) \approx \mathbb{Z}_2$. The only possibly nontrivial differential entering E_{02} is $d^3: E_{30}^3 \rightarrow E_{02}^2$. Here $E_{30}^3 = E_{30}^2 = H_3(\mathbb{Z}_2; \{H_2(S_2 \times S_2)\}) = H_3(\mathbb{Z}_2) \approx \mathbb{Z}_2$. We claim that this d^3 is zero.

To see this consider the spectral sequence $\bar{E}_{ij}^2$ for the subsequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 1,$$

where the subgroup is generated by (12)(34) and the quotient by (1324). Then $\bar{E}_{02}^2 = H_2(\mathbb{Z}_2) = 0$ and $\bar{E}_{30}^3 = \bar{E}_{30}^2 = H_3(\mathbb{Z}_2) = E_{30}^2 = E_{30}^3$. Then naturality of the spectral

sequence implies $d^3 = 0$ as required:

$$\begin{array}{ccc} E_{30}^3 & \xrightarrow{d^3} & E_{02}^3 \\ \approx \uparrow & & \uparrow \\ \bar{E}_{30}^3 & \xrightarrow{\bar{d}^3} & \bar{E}_{02}^3 = 0. \quad \square \end{array}$$

Now it follows that $H_2(S_4)$ is isomorphic to a subgroup of $H_2(G_2) = \mathbb{Z}_2$, since $H_2(S_4)$ is 2-primary. On the other hand, if $H_2(S_4) = 0$ then $H^2(S_4; \mathbb{Z}_2) \approx \mathbb{Z}_2$, contradicting Lemma 2.5.

Corollary 2.9. *The inclusion $S_2 \times S_2 \subset S_4$ induces an isomorphism $H_2(S_2 \times S_2) \rightarrow H_2(S_4)$. The inclusion $S_4 \subset S_n$ ($n \geq 4$) induces an injection $H_2(S_4) \rightarrow H_2(S_n)$.*

Proof. By Lemma 2.8 $H_2(S_2 \times S_2) \rightarrow H_2(G_2)$ is an isomorphism. The map $i: H_2(G_2) \rightarrow H_2(S_4)$ is a map of $\mathbb{Z}_2$ to $\mathbb{Z}_2$. We show it is a surjection. Consider the transfer map $\tau: H_2(S_4) \rightarrow H_2(G_2)$. The map $i \circ \tau$ is multiplication by $[S_4: G_2]$ which is odd. Hence $i \circ \tau$ is an isomorphism, implying i is surjective.

To show that $H_2(S_4) \rightarrow H_2(S_n)$ is injective, it is enough to show that $H_2(S_4) \rightarrow H_2(S_n; \mathbb{Z}_2)$ is injective. This map factors as $H_2(S_4) \rightarrow H_2(S_4; \mathbb{Z}_2) \rightarrow H_2(S_n; \mathbb{Z}_2)$. Since $H_2(S_4) = \mathbb{Z}_2$, the first of these maps is certainly injective. Proving $H_2(S_4; \mathbb{Z}_2) \rightarrow H_2(S_n; \mathbb{Z}_2)$ is injective is equivalent to showing that $H^2(S_n; \mathbb{Z}_2) \rightarrow H^2(S_4; \mathbb{Z}_2)$ is surjective. Now $H^2(S_4; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and maps onto a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ contained in $H^2(S_2 \times S_2; \mathbb{Z}_2)$ by Lemma 2.5. Since the map $H^2(S_n; \mathbb{Z}_2) \rightarrow H^2(S_2 \times S_2; \mathbb{Z}_2)$ is onto the same $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and factors through $H^2(S_4; \mathbb{Z}_2)$ it follows that $H^2(S_n; \mathbb{Z}_2) \rightarrow H^2(S_4; \mathbb{Z}_2)$ is onto.

This completes our sketch for Proposition 2.4 (i). It remains to discuss part (ii).

For $6 \leq n \leq 8$, a 3-Sylow subgroup of A_n is given by the standard $A_3 \times A_3$ generated by (123) and (456). We wish to show that the product map $H_1(A_3) \otimes H_1(A_3) \rightarrow H_2(A_n)$ is injective for $n = 6, 7$ and 0 for $n \geq 8$.

Now $H_1(A_n) \approx \mathbb{Z}_3$ for $n = 3$ or 4 and is zero otherwise. If $n \geq 8$ then the product map above factors through $H_1(A_3) \otimes H_1(A_5) = 0$. Moreover, the commutative diagram

$$\begin{array}{ccc} H_1(A_3) \otimes H_1(A_3) & & H_2(A_7) \\ & \searrow \quad \swarrow & \\ & H_2(A_6) & \end{array}$$

reduces one to the case $n = 7$. The cross product $H_1(A_3) \otimes H_1(A_3) \rightarrow H_2(A_3 \times A_3)$ is an isomorphism by the Künneth formula. So we must show $i_*: H_2(A_3 \times A_3) \rightarrow H_2(A_7)$ is injective. Since $(A_7: A_3 \times A_3)$ is prime to 3, a standard transfer argument says that i_* maps onto the 3-primary part of $H_2(A_7)$, which is thus 0 or $\mathbb{Z}_3$. Therefore it suffices to see that $H_2(A_7; \mathbb{Z}_3) \neq 0$, or equivalently that $H^2(A_7; \mathbb{Z}_3) \neq 0$.

Now by (2.7) $H^2(A_7; \mathbb{Z}_3) \approx H^2(A_3 \times A_3; \mathbb{Z}_3)^{A_7}$. Let x and y denote generators for $H^1(A_3 \times A_3; \mathbb{Z}_3)$ corresponding to the two factors, dual to (123) and to (456). Then $H^2(A_3 \times A_3; \mathbb{Z}_3)$ is $(\mathbb{Z}_3)^3$ generated by the Bocksteins βx and βy and the cup product xy .

It remains to show that xy is an A_7 invariant class. Let $H = A_3 \times A_3$. We must analyze the restriction of xy to $H \cap gHg^{-1}$, as g ranges over A_7 . If $H \cap gHg^{-1} = \{1\}$, then clearly $\text{res}(xy) = \text{res}(g^*(xy)) = 0$. If $H \cap gHg^{-1}$ is cyclic it must be generated by (123) or (456). (It can't contain $(123)^{\pm 1}(456)^{\pm 1}$, because the *only* way to write (123)(456) as a product of two 3-cycles is the obvious one.) Then naturality of cup product shows that again xy and $g^*(xy)$ restrict to 0 on $H \cap gHg^{-1}$.

Finally consider the case $gHg^{-1} = H$. Since g is an even permutation, either g acts as the identity on H or $g(123)g^{-1} = (132)$ and $g(456)g^{-1} = (465)$. In the former case $g^*(xy) = g^*(x)g^*(y) = xy$; in the latter case $g^*(xy) = (-x)(-y) = xy$. So xy is an invariant class as required.

There is one further possibility, namely that $g(123)g^{-1} = (456)^\epsilon$ while $g(456)g^{-1} = (123)^{-\epsilon}$, where $\epsilon = \pm 1$. In this case $g^*(xy) = \epsilon y(-\epsilon)x = -yx = xy$. $\square$

References

- [1] G. Boccara, Décomposition d'une permutation d'une ensemble fini en product de deux cycles. *Discrete Math.* 23 (1978), 189-205.
- [2] K. Brown, *Cohomology of Groups*, Springer-Verlag, New York, Heidelberg, Berlin 1982.
- [3] R.H. Fox, *A Quick Trip Through Knot Theory*, Topology of 3-manifolds, Prentice-Hall, 1962.
- [4] R. Kirby, Problems in low-dimensional manifold theory, *Proc. Symposia in Pure Math.* 32 (1978), 273-312.
- [5] M. Nakaoka, Decomposition theorem for homology groups of symmetric groups, *Annals of Math.* (2) 71 (1960), 16-42.
- [6] J. Roberts, *Elementary Number Theory*, MIT Press, 1977, 153-154.
- [7] D. Rolfsen, *Knots and Links*, Publish or Perish, Berkeley, California, 1976.
- [8] I. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.* 139 (1911), 155-250.
- [9] F. González-Acuna, Homomorphs of knot groups, *Annals of Math.* (2) 102 (1975), 373-377.
- [10] D. Johnson, Homomorphs of knot groups, *Proc. A.M.S.* 78 (1980), No. 1, 135-138.